Foster Reactance Theorem in Fractional Dual Guiding Structures

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Abstract—We have addressed the issue of Foster reactance theorem for fractional dual guiding structures. It is found that for each value of fractional parameter, Foster reactance theorem is satisfied. That is for each value of fractional parameter, fractional dual reactance and fractional dual susceptance are increasing function of frequency.

Index Terms—Fractional curl operator, Fractional dual solution, Foster’s theorem

I. INTRODUCTION

Fractional calculus is a branch of mathematics that deals with operators having non-integer and/or complex order, e.g., fractional derivative and fractional integral [1]. Tools of fractional calculus have various applications in different disciplines of science and engineering, e.g., Optics, Control and Mechanics etc. Discussion on recent applications of tools of fractional calculus in science and engineering is available in [2]. Mathematical recipe to fractionalize a linear operator is available in [3-4]. While exploring the roles and applications of fractional calculus in electromagnetics, a new fractional operator has been introduced [3]. The new fractional operator is termed as fractional curl operator. Fractional curl operator has been utilized to find the new set of solutions to Maxwell equations by fractionalizing the principle of duality [3]. New set of solutions is named as fractional dual solutions to the Maxwell equations. In electromagnetics, principle of duality states that if (E, ηH) is one set of solutions (original solutions) to Maxwell equations, then other set of solutions (dual to the original solutions) is (ηH, −E), where η is the impedance of the medium. The solutions which may be regarded as intermediate step between the original and dual to the original solutions may be obtained using the following relations [3]

\[ E_{fd} = \frac{1}{(jk)^\alpha} (\nabla \times)^\alpha E \]  
\[ \eta H_{fd} = \frac{1}{(jk)^\alpha} (\nabla \times)^\alpha \eta H \]

where \((\nabla \times)^\alpha\) means fractional curl operator and \(k = \omega \sqrt{\mu \varepsilon}\) is the wavenumber of the medium. It may be noted that \(fd\) means fractional dual solutions. It is obvious from above set of equations that for

\[ \alpha = 0, \quad E_{fd} = E, \quad \eta H_{fd} = \eta H \]

and for

\[ \alpha = 1, \quad E_{fd} = \eta H, \quad \eta H_{fd} = -E \]

Which are two sets of solutions to Maxwell equations. The solutions which may be regarded intermediate step between the above two sets of solutions may be obtained by varying parameter \(\alpha\) between zero and one. It may be noted that fractional dual fields represented by \(E_{fd}\) and \(\eta H_{fd}\) are function of fractional parameter \(\alpha\) and the space coordinates. Due to this reason, through out the text each quantity with subscript \(fd\) is function of fractional parameter \(\alpha\) and space coordinates.

Various problems involving fractional curl operator have been studied. These include both wave propagation as well as wave guiding problems. Naqvi et al. [5] extended the work [3] and discussed the behavior of fractional dual solutions in an unbounded chiral medium. Lakhtakia [6] derived theorem which shows that a dyadic operator which commutes with curl operator can be used to find new solutions of the Faraday and Ampere-Maxwell equations. Veliev and Engheta [7] utilized the fractional curl operator for a fixed solution and obtained the fractional fields that represent the solution of reflection problem from an anisotropic surface. Naqvi and Abbas studied the behavior of fractional curl operator for complex and higher orders [8] and fractional dual solutions for metamaterial having negative permittivity and permeability [9]. Naqvi and Rizvi determined the sources corresponding to fractional dual solutions [10]. Recently Hussain and Naqvi [11], introduced the idea of fractional transmission lines and Naqvi et. al. [12] introduced fractional dual waveguides. Naqvi et al. modelled the transmission through chiral layer using fractional curl operator [13].

In present discussion, we proved that Foster’s reactance theorem is valid for fractional dual guiding structures. Term fractional dual guiding structures mean structures which may be regarded as intermediate step of the two given structures which are related through the principle of duality. Dual structure to a given structure may be obtained by applying duality principle to available field solutions for the given structure. Applications of principle of duality reduces a short circuit transmission line to an open circuit line and a waveguide with PEC walls to a waveguide with PMC walls, for example [11-12].

II. ONE PORT JUNCTION

It is well known that for a one port reactive lossless termination of a microwave network, the input impedance \(Z_{in}\) of such a one port termination is purely imaginary, that is \(Z_{in} = jX_{in}\) (and of course \(Y_{in} = jB_{in}\)), and thus it is purely

\[ Z_{in} = jX_{in} \]
reactive. (Here the time dependence of \(\exp(j\omega t)\) is assumed). Foster’s reactance theorem states that in general for such a one port reactive termination [14], we have \(\frac{\partial X_{\text{in}}}{\partial \omega} > 0\) and \(\frac{\partial B_{\text{in}}}{\partial \omega} > 0\). As mentioned in [14], this implies that the poles and zeros of a reactance (or a susceptance) function must alternate along the frequency axis. In current discussion, the issue is to find out whether the fractional dual guiding structures also satisfy Foster’s reactance theorem.

For harmonic fields, Faraday-Ampere Maxwell equations are given by

\[
\nabla \times \mathbf{E} = -j k (\eta \mathbf{H}) \\
\nabla \times (\eta \mathbf{H}) = j k \mathbf{E}
\]

It is obvious that \((\mathbf{E}, \eta \mathbf{H})\) and \((\eta \mathbf{H}, -\mathbf{E})\) are the two dual solutions of Maxwell equations. If \((\mathbf{E}_{fd}, \eta \mathbf{H}_{fd})\) is new solution set which may be regarded as intermediate step between the two dual solutions and may be obtained using (1) and (2). The new solution may be termed as fractional dual solution. Fractional dual solutions and may be obtained using \((1)\) and \((2)\). The new solution set \((\mathbf{E}_{fd}, \eta \mathbf{H}_{fd})\) is given by

\[
\nabla \times \mathbf{E}_{fd} = -j k (\eta \mathbf{H}_{fd}) \\
\nabla \times (\eta \mathbf{H}_{fd}) = j k \mathbf{E}_{fd}
\]

From above two equations, we may write

\[
\nabla \times \frac{\partial \mathbf{E}_{fd}}{\partial \omega} = j k \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} + j \eta \mathbf{H}_{fd} \frac{\partial k}{\partial \omega} \\
\nabla \times \frac{\partial (\eta \mathbf{H}_{fd})}{\partial \omega} = -j k \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} + j \mathbf{E}_{fd} \frac{\partial k}{\partial \omega}
\]

Consider the quantity

\[
\nabla \left( \mathbf{E}_{fd} \times \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} + \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \times \mathbf{H}_{fd} \right) = \nabla \times \mathbf{E}_{fd} \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} - \mathbf{E}_{fd} \nabla \times \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} + \nabla \times \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \mathbf{H}_{fd} - \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \nabla \times \mathbf{H}_{fd}
\]

Substituting from above gives

\[
\nabla \left( \mathbf{E}_{fd} \times \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} + \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \times \mathbf{H}_{fd} \right) = j \left( \mathbf{E}_{fd} \cdot \left( \frac{\partial k}{\partial \omega} + \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \nabla \times \mathbf{H}_{fd} \right) \right)
\]

If we integrate throughout the fractional dual volume of the termination and use the divergence theorem on the left hand side, we obtain

\[
\int_s \left( \mathbf{E}_{fd} \times \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} + \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \times \mathbf{H}_{fd} \right) . ds = -j \int_v \eta \mathbf{H}_{fd} \cdot \frac{\partial k}{\partial \omega} + \mathbf{E}_{fd} \cdot \frac{\partial k}{\partial \omega} dv
\]

where \(W_{mfd} + W_{efd}\) is the total time average energy stored in the lossless volume termination and \(ds\) is chosen directed into the volume.

Since each term in the integrand, vanishes on the surface enclosing volume, the surface integral reduces to an integral over the terminal plane \(t\) only. Terminal plane is any transverse plane at some location in the waveguide. On the terminal plane we have

\[
\int_t \left( \mathbf{E}_{fd} \times \frac{\partial \mathbf{H}_{fd}^*}{\partial \omega} + \frac{\partial \mathbf{E}_{fd}^*}{\partial \omega} \times \mathbf{H}_{fd} \right) . ds = V_{fd} \frac{\partial \mathbf{I}_{fd}^*}{\partial \omega} + \frac{\partial \mathbf{V}_{fd}^*}{\partial \omega} (\eta \mathbf{I}_{fd})
\]

Thus

\[
V_{fd} \frac{\partial \mathbf{I}_{fd}^*}{\partial \omega} + \frac{\partial \mathbf{V}_{fd}^*}{\partial \omega} (\eta \mathbf{I}_{fd}) = -j (\eta \mathbf{I}_{fd}) (\eta \mathbf{I}_{fd}^* \frac{\partial \mathbf{X}_{fd}^*}{\partial \omega})
\]

Hence we obtain

\[
(\eta \mathbf{I}_{fd}) (\eta \mathbf{I}_{fd}^*) \frac{\partial \mathbf{X}_{fd}^*}{\partial \omega} = 4 (W_{efd} + W_{mfd})
\]

The right hand side is proportional to the total energy stored in the fractional volume and can never be negative. Consequently, the slope of the function \(\mathbf{X}_{fd}\) must always be positive.

\[
\frac{\partial \mathbf{X}_{fd}^*}{\partial \omega} = 4 (W_{efd} + W_{mfd}) > 0, \quad 0 < \alpha < 1
\]

Similarly

\[
\frac{\partial \mathbf{B}_{fd}^*}{\partial \omega} = 4 (V_{efd} + V_{mfd}) > 0, \quad 0 < \alpha < 1
\]

III. Conclusion

The result states that for one port reactive termination in fractional dual guiding structure, we have \(\frac{\partial \mathbf{X}_{in}}{\partial \omega} > 0\) and \(\frac{\partial \mathbf{B}_{in}}{\partial \omega} > 0\). This implies that in fractional dual guiding structure for each value of fractional parameter \(\alpha\), the poles and zeros of a fractional dual reactance (or a fractional dual susceptance) function must alternate along the frequency axis.

References


