Modal Characteristics of Quadruple-Clad Planar Waveguides with Double Negative Metamaterials

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Abstract- A theoretical investigation of quadruple-clad dielectric waveguides containing double negative metamaterials but with arbitrary index profiles is presented. Characteristic equations and cutoff conditions for fast-wave and slow-wave modes are derived. This investigation reveals some results with new characteristics not observed in double negative slab waveguides. In particular, it is shown that, unlike double negative slab waveguides in which fast-wave fundamental transverse electric and transverse magnetic modes do not exist, quadruple-clad waveguides containing double negative materials can support these modes. For the case where the inner cladding layers are double negative, the odd and even modes are out of order in a manner not seen in double positive waveguides or even the double negative slab waveguide. Other unique modal characteristics are also observed.

Index Terms- dielectric waveguides, double negative metamaterials, modal characteristics of double negative waveguides.

I. INTRODUCTION

The concept of a material with simultaneously negative real parts of the permittivity and permeability was first investigated by Veselago in 1968 [1]. Although the subject remained dormant for many years, interest was revived with the creation of a metamaterial with double negative characteristics by Smith et al. in 2000 [2]–[4]. Since that time, further analysis of double negative (DNG) metamaterials has been undertaken. Waveguides are one area which has, in theory, displayed some interesting new propagation characteristics due to the inclusion of DNG materials [5]–[11]. Unique propagation properties of waveguides containing DNG materials have inspired researchers to envisage novel device applications for such waveguides [5]. The three-layer planar slab waveguide [6]–[10] and circularly cylindrical fiber [11] with DNG cores have been investigated. For planar geometries, several articles have been published dealing with multiple-layer structures containing DNG metamaterials, including transmission through multiple-slab structures [12]–[17] and propagation in coupled waveguide structures [18]. A symmetric five-layer structure was considered in [19] for transmission and some propagation characteristics, however not exhaustively for the latter. Coupled waveguide structures using DNG materials present interesting possible applications where coupling power in the opposite direction is of value [18].

This paper presents an investigation of quadruple-clad planar dielectric waveguides containing DNG materials. The permittivity and permeability of the core and of the cladding layers can be both negative or both positive. Generalized solutions of the electromagnetic fields and the eigenvalue equation for this geometry are presented. The DNG metamaterials are assumed to be lossy and dispersive by way of a causal material model from the literature. It was found that the inclusion of multiple claddings produces some interesting variations from the results obtained for the DNG slab planar waveguide. It is shown that the fundamental mode, which is absent in DNG slab waveguides [6], reappears for some refractive index profiles. The existence of the fundamental mode in the quadruple-clad waveguides investigated here can significantly enhance the power coupling efficiency from excitation sources into such waveguides. This is due to the fact that the transverse field distribution of the fundamental
mode exhibits a pick on the waveguide plane of symmetry, whereas the second mode, say TE1, includes a null on that plane, making its excitation more difficult and less efficient. Further, it is shown that in quadruple-clad DNG waveguides new modes emerge by adjusting the dimensions of the guide and that even and odd modes may appear out of order for some refractive index profiles.

II. GENERALIZED FIELD SOLUTIONS

The structure of interest is a two-dimensional planar waveguide consisting of a central core and four cladding layers placed symmetrically around it. Fig. 1 shows the geometry and electromagnetic parameters for this waveguide. At least one set of parameters εi, μi; i = 1,2,3, is assumed to be negative. Although the analysis presented here is for a five-layer structure, it can be extended to structures with an arbitrary number of symmetrically added cladding layers. It is assumed that there are no variations in material properties in the y direction, and thus field solutions are independent of the y coordinate.

The electromagnetic fields of guided modes are solutions of Maxwell’s equations subject to boundary conditions. Furthermore, assuming that propagation of guided modes occurs along the positive z-direction, the z-dependence of the fields can be expressed as exp(−jβz), where β is the axial propagation constant. The solutions can be classified as transverse electric (TE) modes with non-zero field components Ex, Hz, and Hy and transverse magnetic (TM) modes with non-zero field components Hy, Ez, and Ex. Moreover, the symmetry of the waveguide structure about the x = 0 plane allows both TE and TM modes to be further divided into even and odd groups. Complete field solutions can be obtained if the axial field components are determined, as transverse field components are readily obtained from the axial components. Also, due to the symmetry of the waveguide, it suffices to write field solutions in the half space x > 0.

The generalized solution for the axial field component is written as

\[ \psi_i(x) = \begin{cases} A_iZ_i(q_1x) + B_i\overline{Z}_i(q_1x), & x < a \\ A_iZ_i(q_2x) + B_i\overline{Z}_i(q_2x), & a < x < b \\ A_i e^{\nu_i x}, & x > b \end{cases} \]  

(1)

where \( \psi_i \) represents \( H_i \) or \( E_i \) for TE or TM modes, respectively. Functions \( Z_i, \overline{Z}_i \), and \( \overline{Z}_i \) are defined in Table 1. Also, the analysis uses the defined parameters \( q_i = k_0 \sqrt{ n_i^2 - \beta^2 } \); \( i = 1,2,3 \), with \( n_i^2 = \mu_i \varepsilon_i \), \( \beta = \beta / k_0 \), \( k_0 = \omega \sqrt{ \varepsilon_0 \mu_0 } \), and

\[ \nu_i = \begin{cases} 1, & n_i^2 > \beta^2 \\ -1, & n_i^2 < \beta^2 \end{cases} \]  

(2)

Table 1: Definitions of functions \( Z_i \) and \( \overline{Z}_i \) for \( i = 1,2 \)

<table>
<thead>
<tr>
<th>( Z_i )</th>
<th>( \overline{Z}_i )</th>
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<tr>
<td>( u_i = 1 )</td>
<td>( u_i = -1 )</td>
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<tr>
<td>( \sin )</td>
<td>( \sinh )</td>
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<td>( \cos )</td>
<td>( \cosh )</td>
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Also, for even (odd) modes the amplitude coefficient \( B_i \) \( A_i \) is set to zero. Imposing the boundary conditions at \( x = a \) and \( x = b \) results in a system of four homogeneous equations with amplitude coefficients as unknowns. The eigenvalue or characteristic equation, from which
the normalized propagation constant \( \beta \) can be calculated, is obtained by setting the determinant of this system of equations equal to zero. Also, from these equations all amplitude coefficients can be determined in terms of one arbitrarily chosen independent coefficient. The results for the amplitude coefficients \( A_2, B_2, \) and \( A_3 \) in terms of \( A_1 \) for TM modes (or \( B_1 \) for TE modes) are summarized in the Appendix. Defining 

\[
U = q_1a, \; X = q_2a, \; Y = q_2b, \; \text{and} \; W = q_3b, \]

the characteristic equation for even TM modes is

\[
U T_i(U) = \frac{v_1 \varepsilon_1}{v_2 \varepsilon_2} X T_2(Q + X - Y) \]  

(3)

where

\[
T_i = \begin{cases} 
\tan, & v_i = 1 \\
\tanh, & v_i = -1 
\end{cases} \]  

(4)

with \( Q = T_2^{-1} \left[ v_2 \varepsilon_2 W / (\varepsilon_1 Y) \right] \) when \( v_2 = 1 \), or \( v_2 = -1 \) and \( |v_2 \varepsilon_2 W / (\varepsilon_1 Y)| < 1 \). For \( v_2 = -1 \) and \( |v_2 \varepsilon_2 W / (\varepsilon_1 Y)| > 1 \), the term \( T_2(Q + X - Y) \) in (3) should then be replaced with \( \coth (\hat{Q} + X - Y) \) where \( \hat{Q} = \coth^{-1} \left[ v_2 \varepsilon_2 W / (\varepsilon_1 Y) \right] \).

The characteristic equations for even TE modes may be found from (3) by replacing \( e_i \) with \( -\mu_i \) for \( i = 1, 2, 3 \) in (3) (including \( Q \) or \( \hat{Q} \)). Also, the characteristic equations for odd modes may be obtained from those for the corresponding even modes by replacing the term \( T_i(U) \) with \(-v_i / T_i(U)\). The characteristic equations for DNG slab waveguide are contained in those of the quadruple-clad waveguide as a special case in the limit of \( a \to b \). For example, by setting \( X = Y \) and \( v_1 = 1 \), (3) reduces to \( U \tan(U) = (e_i / \varepsilon_1)W \) which is the characteristic equation for even TM modes in a slab waveguide.

III. FAST-WAVE AND SLOW-WAVE MODES

It may be shown analytically that for guided modes in waveguides containing double positive materials the solutions are restricted to \( n_3 < \beta < max(n_1, n_2) \). Numerical analysis lends evidence to this restriction, as no solutions are seen above the maximum of \( n_1 \) or \( n_2 \). However, it may likewise be shown that when DNG materials are included, the upper limit on the solutions disappears such that \( |n_3| < \beta \) is the only condition. Numerical analysis also supports this by revealing the presence of solutions for \( \beta \) greater than the maximum of \( |n_1| \) and \( |n_2| \).

When \( \beta < max(|n_1|, |n_2|) \) the modes are referred to as ‘fast-wave’ modes, as the phase velocity of these modes, defined as \( v_p = \omega / \beta \), is greater than that in the infinite material space with the larger of the refractive indices \( |n_1| \) or \( |n_2| \). On the other hand, slow-wave modes correspond to \( \beta > |n_i| \) for all \( i \), and thus phase velocity is less than that of the infinite material space with \( n = max(|n_1|, |n_2|, |n_3|) \). Waveguides with double positive materials support only fast-wave modes, whereas the inclusion of DNG materials allows for slow-wave modes as well.

For fast-wave modes, at least one of the \( T_i \) or \( T_2 \) functions in (3) assumes the periodic ‘\( \tan \)’ form implying that the characteristic equation at a given frequency may possess multiple solutions, each corresponding to a mode. However, for slow-wave modes, both the \( T_i \) and \( T_2 \) functions in (3) assume the non-periodic ‘\( \tanh \)’ form, suggesting a limited number of solutions. The specific number of solutions (zero, one, or two) is determined by the material models used and the parameters of the waveguide. Thus, while many fast-wave-modes may be supported by the waveguide, a finite number of slow-wave modes exist. Further insight into the precise number of available slow-wave modes may be acquired by examining the cutoff conditions of the five-layer configuration.
The cutoff frequencies of fast-wave modes can be found from the characteristic equations by setting \( \beta = |n_3| \), or equivalently \( W = Q = 0 \), if \( |n_3| < |n_1| \) or \( |n_2| \). In doing so, the cutoff condition for fast-wave even TM modes is obtained from (3) as

\[
U T_1(U) = \frac{V_1}{V_2} \frac{\varepsilon_1}{\varepsilon_2} X T_2(X - Y) .
\]  

(5)

Cutoff conditions for fast-wave even TE as well as those for odd TM and TE modes are obtained from (5) by making the same changes described earlier for the characteristic equation.

Cutoff frequencies for slow-wave modes may be obtained from their respective characteristic equations by setting \( \beta = |n_1| \) (or \( U = 0 \)) if \( |n_1| > |n_2|, |n_3|, \), \( \beta = |n_2| \) (or \( X = Y = 0 \)) if \( |n_2| > |n_1|, |n_3| \), or \( \beta = |n_3| \) (or \( W = Q = 0 \)) if \( |n_3| > |n_1|, |n_2| \). For the first case, the cutoff frequencies for even TM modes may be found using

\[
\tanh^{-1} \left( \frac{\varepsilon_2 W}{\varepsilon_3 Y} \right) = X - Y .
\]  

(6)

For the odd case, the cutoff equation is

\[
\varepsilon_1 X \tanh(Q + X - Y) - \varepsilon_2 = 0 .
\]  

(7)

The TE cases may be found using the same transformation rules as the characteristic equation. Since the left-hand side of (6) is always negative, solutions only exist when either the inner or outer cladding layers are DNG (and thus either \( \varepsilon_2 \) or \( \varepsilon_3 \) is negative). For the second case where \( \beta = |n_3| \) the cutoff frequencies for the slow-wave even TM modes are obtained from

\[
U \tanh U + \frac{\varepsilon_1}{b (\varepsilon_2 + \varepsilon_3 \frac{1}{W}) - \varepsilon_2} = 0 .
\]  

(8)

Once again, the usual transformations to find odd and TE modes apply. In the third and final case where \( \beta = |n_3| \) the cutoff frequencies are found using a reduced form of (5).

\[
U \tanh(U) = \frac{\varepsilon_1}{\varepsilon_2} X \tanh(X - Y) .
\]  

(9)

The fundamental fast-wave TM\(_0\) and TE\(_0\) modes do not exist in DNG slab waveguides, as noted from the characteristic equations of even modes in such guides: \( U \tan U = (\varepsilon_1 / \varepsilon_2)W \) for TM and \( U \tan U = (\mu_1 / \mu_2)W \) for TE modes. When \( \varepsilon_1 \) and \( \mu_1 \) are negative the first root of the characteristic equations occurs for \( U > \pi / 2 \), whereas for the fundamental modes of double positive slab waveguide \( U < \pi / 2 \). However, in quadruple-clad waveguides examination of (5) reveals that when \( \varepsilon_1 \) or \( \varepsilon_2 \) is negative it is possible to have a root corresponding to the fundamental TM\(_0\) mode. Similarly, when \( \mu_1 \) or \( \mu_2 \) is negative it is possible to have a root corresponding to the fundamental TE\(_0\) mode. Numerical results presented in the next section reaffirm the existence of these modes.

**IV. NUMERICAL ANALYSIS**

As Veselago predicted, and as subsequent analysis has further demonstrated, DNG materials are inherently dispersive. As a result, it is important to use a causal, dispersive material model in order to obtain realistic results for dielectric waveguides containing these materials. One possibility is the double Lorentzian model where the permittivity \( \varepsilon \) and the permeability \( \mu \) are given by (A.4) and (A.5), respectively, in the Appendix. This model has been used in the literature to describe an actual DNG metamaterial [3], [4]. The parameters from [3] shall be used here and are given as \( \omega_{ep} = 80.42 \text{GHz}, \omega_{eo} = 64.72 \text{GHz}, \omega_{mp} = 68.8 \text{GHz}, \omega_{m0} = 63.15 \text{GHz} \) and \( \gamma = 10 \text{MHz} \). Since this model contains the imaginary components of the material parameters, thus including material loss, by invoking perturbation theory it can be noted that the imaginary components may be ignored when they are small compared to the real components. Loss may be
considered later as a perturbation of the solution. The conditions for this approach are in fact met over most of the frequency range of interest, except at the low and high ends. The material model of (A.4) and (A.5) has DNG characteristics over the frequency range of approximately 64.7 to 68.8 GHz. Fig. 2 shows a plot of the ratio of the real to imaginary components of the material parameters with respect to the chosen level of 100, which shall serve as the approximate minimum ratio for the applicability of the perturbation method.

Fig. 3 through Fig. 6 depict the TE and TM dispersion characteristics of a five-layer waveguide with \( a = 0.5 \) cm, \( b = 1 \) cm, and the various possible refractive index profiles where either the core or inner cladding layers are DNG. The plots are shown in terms of \( \beta \) versus frequency \( \omega \). It is noteworthy that the group velocity of the solutions may be very small or negative, as shown in Fig. 7. These results show the group velocity, as calculated using the dispersion curves, for the first few even TE modes from Fig. 3. Double degeneracy is also observed in some of the modes. These results have been seen in other planar DNG waveguides (e.g., [20]).
Fig. 4. The TE (a) and TM (b) dispersion characteristics (solid) for several modes in a five-layer waveguide with $a = 0.5$ cm, $b = 1$ cm, $\varepsilon_3 = 2.0 \varepsilon_0$, $\mu_3 = \mu_0$, $\varepsilon_2 = \varepsilon_0$, $\mu_2 = \mu_0$ and $\varepsilon_1$ and $\mu_1$ governed by (A.4) and (A.5) respectively. The maximum refractive index is also shown (dashed) which delineates between slow-wave (above) and fast-wave (below) modes. The absolute values of the refractive index profile is inset (where $n_1$ varies with frequency). The fast-wave fundamental mode is absent in both cases.

Fig. 5. The TE (a) and TM (b) dispersion characteristics (solid) for several modes in a five-layer waveguide with $a = 0.5$ cm, $b = 1$ cm, $\varepsilon_3 = \varepsilon_0$, $\mu_3 = \mu_0$, $\varepsilon_1 = 2.0 \varepsilon_0$, $\mu_1 = \mu_0$ and $\varepsilon_2$ and $\mu_2$ governed by (A.4) and (A.5) respectively. The maximum refractive index is also shown (dashed) which delineates between slow-wave (above) and fast-wave (below) modes. The absolute value of the refractive index profile is inset (where $n_2$ varies with frequency). The fast-wave fundamental mode is present in both cases.
Fig. 6. The TE (a) and TM (b) dispersion characteristics (solid) for several modes in a five-layer waveguide with $a = 0.5$ cm, $b = 1$ cm, $\varepsilon_1 = \varepsilon_2$, $\mu_2 = \mu_0$, $\varepsilon_3 = 2.0\varepsilon_0$, $\mu_4 = \mu_0$ and $\varepsilon_2$ and $\mu_2$ governed by (A.4) and (A.5) respectively. The maximum refractive index is also shown (dashed) which delineates between slow-wave (above) and fast-wave (below) modes. The absolute value of the refractive index profile is inset (where $n_2$ varies with frequency). The fast-wave fundamental mode is absent in both cases. Also, two modes are in fact present in each TM curve but are difficult to distinguish due to near overlapping.

Fig. 7. The group velocity, expressed as a fraction of the speed of light $c$, for the first three even TE modes (solid) for the waveguide parameters of Fig. 3. Zero velocity is demarcated for reference (dashed). Portions of some modes exhibit negative group velocity.

One noteworthy result from the numerical analysis is that the presence of the fundamental fast-wave mode, as predicted analytically, is confirmed for some refractive index profiles, such as in Fig. 3. Furthermore, the specific number of slow-wave modes is determined by the waveguide parameters and ranges from zero to four total modes. Another interesting result is the absence of both the TM$_0$ and the TM$_1$ modes in Fig. 4. This is a unique result, as these modes are not just missing from the fast-wave hierarchy but from both fast- and slow-wave hierarchies.

By far the most striking results are seen for the case where the inner cladding layers are DNG (Fig. 5 and Fig. 6). In these cases, the order of the even and odd modes is in fact reversed (with the exception of the TM modes in Fig. 5, where only the higher order even and odd modes are reversed, and the TM$_3$ mode is missing altogether). Another interesting feature can be noted in the fact that the even and odd modes are extremely close together, whereas there is a more even spacing in other cases. Furthermore, whereas in the case of the core being DNG there are a maximum of two slow-wave modes for either TE or TM, in this case there can be up to four slow-wave modes (in these examples, only for TE). It can be seen, then, that the inclusion of a DNG inner cladding in a five-layer planar waveguide leads to many new characteristics not seen in other waveguides.

Interestingly, the number of modes may in fact be increased by increasing the thickness of the inner cladding (i.e., by increasing $b$). This is as opposed to an increase in mode density, since new modes with the same number of nodes in the transverse field distribution as already existing modes begin to appear at the high-frequency end of the DNG range. Fig. 8 illustrates this with dispersion plots for several values of $b$. The number of new modes may be made arbitrarily
large through a sufficient increase in $b$. Similar to the case of DNG inner claddings, the new modes appear out of order with respect to even and odd solutions. First the higher-order odd mode appears followed by the associated even mode of lower order.

In order to deal with loss associated with the material models based on (A.4) and (A.5) and employed in this analysis, it is sufficient to invoke perturbation theory after the lossless analysis such as was performed above. Loss is manifested as exponential decay of the fields with increasing distance from the source (positive $z$ direction) and has the form $\exp(-\alpha z)$ where $\alpha$ must be positive for decay. This loss parameter, $\alpha$, may be calculated according to (A.6). Fig. 9 shows a comparison of results for $\alpha$ as calculated exactly using a complex root search of the eigenvalue equation and by the perturbation theory for one mode.

V. CONCLUSION

The quadruple-clad planar dielectric waveguide was examined in detail. The analysis revealed that DNG waveguides with more than three layers do in fact allow for the fundamental fast-wave mode in at least one case. This mode was found to be absent in three-layer waveguides containing DNG materials [6]. Further unique modal characteristics were seen, including the complete absence of certain low order modes along with a reversal of the expected order of even and odd modes for some configurations. New modes may be seen to appear as the waveguide parameters are adjusted, this being another unique property of quadruple-clad DNG waveguides not seen for the simple DNG slab. These interesting results suggest that the five-layer waveguide with DNG inner claddings (i.e., a DNG "tube"), especially, has characteristics which cannot be duplicated with double-clad waveguides or with double positive waveguides of any kind.

APPENDIX

A. Expressions for Amplitude Coefficients

$$A_2 = A_1[\xi Z_1'(U)\bar{Z}_1(X) - Z_1(U)\bar{Z}_1'(X)]$$  \(\text{(A.1)}\)

$$B_2 = A_1[Z_1(U)\bar{Z}_1'(X) - \xi Z_1'(U)\bar{Z}_1(X)]$$  \(\text{(A.2)}\)

$$A_4 = A_1e^{\xi \phi}[\xi Z_1'(U)Z_2(Y-X) + Z_1(U)\bar{Z}_2'(Y-X)]$$  \(\text{(A.3)}\)
where $Z_i$ and $Z'_i$ are defined in Table 1, $U = q_a$, $X = q_a$, $Y = q_b$, and $W = q_b$, and $\xi = v_\varepsilon X / (v_\varepsilon U)$.

The above expressions are for even TM modes. The amplitude coefficients for even TE modes may be obtained from (A.1) to (A.3) by substituting $A_i$ with $B_i$ and $\xi$ with $\xi = v_\varepsilon X / (v_\varepsilon U)$.

Furthermore, the amplitude coefficients for odd modes may be obtained from the amplitude expressions for the corresponding even modes in which $Z_i(U)$ and $Z'_i(U)$ are replaced with $\bar{Z}_i(U)$ and $\bar{Z}'_i(U)$, respectively.

B. Double Lorentzian Model for $\varepsilon$ and $\mu$

$$\varepsilon_r(\omega) = 1 - \frac{\omega_{ep}^2 - \omega_{eo}^2}{\omega^2 - \omega_{eo}^2 - j \gamma \omega} \quad (A.4)$$

$$\mu_r(\omega) = 1 - \frac{\omega_{mp}^2 - \omega_{mo}^2}{\omega^2 - \omega_{mo}^2 - j \gamma \omega} \quad (A.5)$$

C. Loss Parameter $\alpha$

$$\alpha = \frac{1}{2} \int \frac{\left( H^* \cdot \hat{E} + \varepsilon'' |\hat{E}|^2 \right) dx}{Re \int \hat{E} \times \hat{H}^* \cdot \hat{z} dx} \quad (A.6)$$

where the imaginary components of the material parameters are $\mu''$ and $\varepsilon''$ [21].

REFERENCES


