Semi-analytical Approach for Microstructured Optical Fibers Using a Model of Layered Cylindrical Arrays

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Abstract- A rigorous semi-analytical approach for specific microstructured optical fibers, which are formed by layered cylindrical arrays of circular rods symmetrically distributed on each of concentric circular rings, is presented for both of scalar-wave formulation and full-wave formulation. The method uses the T-matrix of a single circular rod, the reflection and transmission matrices for a cylindrical array, and the generalized reflection and transmission matrices of a cylindrically layered system.

Index Terms- semi-analytical approach, cylindrical arrays of circular rods, microstructured optical fibers

I. INTRODUCTION

Photonic crystal fiber (PCF) is a new class of optical fibers with unique confinement characteristics, which are not possible in conventional optical fibers. PCF with a core of a higher average index than the microstructured cladding is a topic of active research because of its potential importance in diverse areas such as fiber-optic communications, fiber lasers and amplifiers including high-power devices, nonlinear photonic devices, and highly sensitive sensors [1][2]. The modal properties of photonic crystal fibers have been extensively investigated using the finite element method [3], the finite difference frequency domain method [4], and the multipole method [5]. These methods could be versatilely applied to various microstructured configurations but they are computationally intensive. Recently, approximate analytical methods using the effective index model [6][7] or a variational method [8] have been proposed to investigate the fundamental modal properties of photonic crystal fibers. However, the range of applicability of the approximation in the methods is not so clear.

In this paper, we present a rigorous semi-analytical approach for the modal analysis of specific microstructured fibers, which consist of layered cylindrical arrays of circular rods symmetrically distributed on each of concentric circular rings. The method uses the T-matrix of a circular rod in isolation, the reflection and transmission matrices for a cylindrical array, and the generalized reflection and transmission matrices for a cylindrically layered structure [9][10]. Both of the scalar-wave formulation and the vector-wave formulation are discussed.

II. GEOMETRY OF THE PROBLEM

The cross sectional view of a specific microstructured fiber, which is formed by \( N \)-layered cylindrical arrays of circular rods located in a homogeneous background medium with material constants \( \varepsilon_r \) and \( \mu_r \), is shown in Fig. 1. The \( M \) circular rods are symmetrically distributed on each of the \( N \)-concentric circular rings with radii \( R_1, R_2, \ldots, R_N \). The \( M \) circular rods should be identical along one ring but those on different rings need not be necessarily identical. The circular ring with radius \( R_{\nu} \) is labeled as the \( \nu \)-th layer. The radius and material constants of the circular rods on the \( \nu \)-th layer are denoted by \( r_{\nu}, \varepsilon_{\nu}, \) and \( \mu_{\nu} \), respectively. The concentric homogeneous region within \( R_{\nu} < \rho < R_{\nu+1} \) is labeled as the \((\nu)\)-th region.
III. SCALAR-WAVE FORMULATION

A. Field Representation

A scalar wave analysis is employed. From the symmetry of the structure, the field is not changed under the rotation of the coordinate by an angle of $\theta_M = 2\pi / M$ with respect to the global origin $O$. Then the field in the $(\nu)$-th region can be expressed in the following form:

$$
\psi^{(\nu)} = e^{i\beta z} \sum_{m=-\infty}^{\infty} b_m^{(\nu)} J_m(\kappa r) e^{im\phi} + c_m^{(\nu)} Y_m(\kappa r) e^{im\phi} 
$$

(1)

where $\kappa = \sqrt{n_e^2 k_0^2 - \beta^2}$, $k_0 = 2\pi / \lambda_0$, $\lambda_0$ is the wavelength in free space, $n_e = \sqrt{\varepsilon_\mu}$ is the refractive index of the background medium, $J_m$ and $Y_m$ are the Bessel and Neumann functions of the $m$-th order, respectively, $(b_m^{(\nu)})$ represent the amplitudes of the incoming standing cylindrical harmonics scattered from the outer circular rods on the $(\nu-1)$-th layer, $(c_m^{(\nu)})$ are those of outgoing cylindrical harmonics scattered from the inner circular rods on the $\nu$-th layer, $\beta$ is the propagation constant. Equation (1) is rewritten in the matrix form as follows:

$$
\psi^{(\nu)} = e^{i\beta z} [\Phi^T \cdot b^{(\nu)} + \Psi^T \cdot c^{(\nu)}] 
$$

(2)

with

$$
\Phi = [J_m(\kappa h) e^{im\phi}] \quad (3)$$

$$
\Psi = [Y_m(\kappa h) e^{im\phi}] \quad (4)$$

$$
\mathbf{b}^{(\nu)} = [b_m^{(\nu)}], \quad \mathbf{c}^{(\nu)} = [c_m^{(\nu)}] \quad (5)
$$

where the vector quantities are defined as the column vectors and the superscript $T$ denotes the transpose of the indicated vectors.

B. Reflection and Transmission Matrices

The scattering process to characterize the reflection and transmission of the $\nu$-th cylindrical array is schematically illustrated in Fig. 2. Firstly, we consider the case where the incoming field $\mathbf{b}^{(\nu)}$ is incident from the $(\nu)$-th region on the $\nu$-th layer and scattered. The scattered field in the region of $r_j > r_c$ ($j = 1, 2, \cdots, M$) is expressed as follows:

$$
\psi^{(\nu)} = \sum_{j=1}^{M} \sum_{m=-\infty}^{\infty} a_m^{(\nu)} Y_m(\kappa r_j) e^{im\phi_j} 
$$

(6)
where \((\rho_0^j, \varphi_0^j)\) is the local cylindrical coordinate system whose origin is located at the center of the \(j\)-th circular rod as shown in Fig. 2, \(\{a_m^{(v)}\}\) denote the unknown scattering amplitudes. It should be noted that \(\{a_m^{(v)}\}\) take the same values for all the circular rods on the \(v\)-th layer because of the symmetry of the structure. Equation (6) can be rewritten in the matrix form as follows:

\[
\psi^{(v)} = e^{i\theta z} \sum_{m=1}^{M} \psi_m^{(v)T} \cdot a^{(v)}
\]  

(7)

with

\[
\psi_m^{(v)} = [Y_m(\kappa \rho^j) e^{i m \varphi^j}]
\]  

(8)

\[
a^{(v)} = [a_m^{(v)}].
\]  

(9)

The scattered field (7) must satisfy the boundary conditions on the surfaces \(\rho^j = r_c\) for all the circular rods on the \(v\)-th layer under the presence of the incident field \(\Phi^T \cdot b^{(v)}\). Because of the symmetry of the structure, we can consider the boundary condition only on \#1 rod in Fig. 2.

In order to apply the boundary condition on \#1 circular rod, the scattered field components based on the local coordinates \((\rho^j, \varphi^j)\) are transformed into those expressed in terms of the local coordinate \((\rho^1, \varphi^1)\). This transformation is easily performed using Graf’s addition theorem for the cylindrical functions. After straightforward manipulations, from (7) we have the following relation:

\[
\psi^{(v)}(\rho^1, \varphi^1) = \psi_1^{(v)T} \cdot a^{(v)} + \psi_1^{(v)T} \cdot K^{(v)} \cdot a^{(v)}
\]  

(10)

with

\[
\psi_1^{(v)} = [J_\nu(\kappa \rho_1) e^{i m \varphi_1}]
\]  

(11)

\[
K^{(v)} = \sum_{j=2}^{M} w_j^{(v)}
\]  

(12)

\[
w_j^{(v)} = [Y_m(\kappa \rho^j) e^{i m \varphi^j}]\]

(13)

\[
\zeta_j = \pi / 2 - (j - 1) \theta_N / 2
\]  

(14)

\[
d_j^r = 2R_c \sin[(j-1) \theta_d / 2]
\]  

(15)

where \(w_j^{(v)}\) is the translation matrix of the singular part of cylindrical harmonics from the \(j\)-th local coordinate to the \#1 local coordinate, and \(d_j^r\) is the distance between the centers of the \# \(j\) and \# \(1\) rods.

In the second step of the analysis, the incident field \(\Phi^T \cdot b^{(v)}\) given in the global coordinate \((\rho, \varphi)\) is transformed into the local coordinate \((\rho^1, \varphi^1)\) using Graf’s addition theorem as

\[
\Phi^T \cdot b^{(v)} = \Phi_1^{(v)T} \cdot a^{(v)} \cdot b^{(v)}
\]  

(16)

with

\[
a^{(v)} = [a_m^{(v)}], \quad a^{(v)}_\alpha = (-1)^{M-\alpha} J_{M-\alpha}(\kappa r_c)
\]  

(17)

where \(a^{(v)}\) is the translation matrix of the regular part of cylindrical harmonics from the global coordinate to the \#1 local coordinate. Using (10) and (16), the total field around \#1 circular rod on the \(v\)-th layer is expressed as

\[
\psi^{(v)}(\rho^1, \varphi^1) = \psi_1^{(v)T} \cdot a^{(v)} + \Phi_1^{(v)T} \cdot (K^{(v)} \cdot a^{(v)} + a^{(v)} \cdot b^{(v)}).
\]  

(18)

In (18), the second term may be viewed as the incident field impinging on \#1 circular rod, whereas the first term is the scattered field from the same rod. This argument leads to a relationship between the amplitude vectors \(b^{(v)}\) and \(a^{(v)}\) as follows:

\[
a^{(v)} = T^{(v)} \cdot (K^{(v)} \cdot a^{(v)} + a^{(v)} \cdot b^{(v)})
\]  

(19)

where \(T^{(v)}\) is the T-matrix of the circular rod on the \(v\)-th layer in isolation [11], whose closed-form expression is given in standard textbooks. From (19) we have

\[
a^{(v)} = \overline{T}_v^{(v)} \cdot b^{(v)}
\]  

(20)

with

\[
\overline{T}_v^{(v)} = (I - T^{(v)} \cdot K^{(v)})^{-1} T^{(v)} \cdot a^{(v)}
\]  

(21)

where \(I\) is the unit matrix. In (20), \(\overline{T}_v^{(v)}\) may be regarded as an aggregate T-matrix of the \(M\) circular rods for the incidence of incoming cylindrical harmonic waves.

Substituting (20) into (7), the scattered field is expressed in terms of the amplitude vector \(b^{(v)}\) of the incident field. However, the expression is still based on the local coordinate system \((\rho^j, \varphi^j)\) \((j = 1, 2, \ldots, M)\) of the circular rods. As the third step, (7) is transformed into the expression based on the global coordinate system \((\rho, \varphi)\) using Graf’s addition theorem. The resulting expressions are different in the \((v)-\)th region with \(\rho > R_c\) and the \((v-1)-\)th region with \(\rho < R_c\). When \(\rho > R_c\), (7) is rewritten as follows:
\[ \psi^{(v)} = \Psi^T \cdot \chi^{(v)} \cdot a^{(v)} \] (22)

where

\[ \chi^{(v)} = [\chi_{mn}^{(v)}], \quad \chi_{mn}^{(v)} = MJ_{m-l-n}(\kappa R_v). \] (23)

For the incident field \( \Phi^T \cdot b^{(v)} \) from the \( v \)-th region, (22) yields the reflected field into the same region. Substituting (20) into (22), the reflected field into the \( v \)-th region may be expressed as follows:

\[ \psi^{(v)} = \Psi^T \cdot R_{\nu,v-1} \cdot b^{(v)} \] (24)

with

\[ R_{\nu,v-1} = \chi^{(v)} \cdot \bar{T}^{(v)} \] (25)

where \( \bar{T}^{(v)} \) is defined by (21). From the comparison of (24) with (2), it follows that \( R_{\nu,v-1} \) gives the reflection matrix of the \( v \)-th layer, which defines the reflection from the \( (v-1) \)-th region to the \( v \)-th region through the circular rods on the \( v \)-th layer.

On the other hand, when \( \rho < R_v \), the scattered field is expressed in global coordinate system \((\rho, \varphi)\) as follows:

\[ \psi^{(v-1)} = \Phi^T \cdot \eta^{(v)} \cdot a^{(v)} \] (26)

where

\[ \eta^{(v)} = [\eta_{mn}^{(v)}], \quad \eta_{mn}^{(v)} = MY_{m-l-n}(\kappa R_v). \] (27)

Substituting (20) into (26), the transmitted field into the \( (v-1) \)-th region is obtained as

\[ \psi^{(v-1)} = \Phi^T \cdot F_{\nu-1,v} \cdot b^{(v)} \] (28)

with

\[ F_{\nu-1,v} = I + \eta^{(v)} \bar{T}^{(v)} \] (29)

where \( F_{\nu-1,v} \) gives the transmission matrix of the \( v \)-th layer, which defines the transmission from the \( (v-1) \)-th region to the \( (v-1) \)-th region through the circular rods on the \( v \)-th layer. Note that the unit matrix \( I \) contained in (29) indicates the contribution of the original incident field \( \Phi^T \cdot b^{(v)} \) in the inner region.

Let us consider next that the field \( \Psi^T \cdot c^{(v-1)} \) is incident from the \( (v-1) \)-th region on the \( v \)-th layer of the circular rods and scattered. Following the same analytical procedure as described above, the transmitted field \( \psi^{(v)} \) into the \( v \)-th region with \( \rho > R_v \) and the reflected field \( \psi^{(v-1)} \) into the \( (v-1) \)-th region with \( \rho < R_v \) are obtained as

\[ \psi^{(v)} = \Psi^T \cdot F_{\nu,v-1} \cdot c^{(v-1)} \] (30)

\[ \psi^{(v-1)} = \Phi^T \cdot R_{\nu-1,v} \cdot c^{(v-1)} \] (31)

with

\[ F_{\nu,v-1} = I + \chi^{(v)} \bar{T}^{(v)} \] (32)

\[ R_{\nu-1,v} = \eta^{(v)} \bar{T}^{(v)} \] (33)

\[ \bar{T}^{(v)} = (I - T^{(v)} K^{(v)})^{-1} T^{(v)} \gamma^{(v)} \] (34)

\[ \gamma^{(v)} = [\gamma_{mn}^{(v)}], \quad \gamma_{mn}^{(v)} = (-1)^{m-l-n} Y_{m-l-n}(\kappa R_v). \] (35)

where \( F_{\nu,v-1} \) defines the transmission matrix from the \( (v-1) \)-th region to the \( v \)-th region and \( R_{\nu-1,v} \) defines the reflection matrix from the \( (v) \)-th region to the \( (v-1) \)-th region. If the cylindrical harmonic expansion in (1) is truncated by \( m \leq L \), \( R_{\nu,v-1} \), \( R_{\nu-1,v} \), \( F_{\nu,v-1} \), and \( F_{\nu-1,v} \) are reduced to the matrices of \((2L+1) \times (2L+1)\) dimensions.

It should be noted that the reflection and transmission matrices are defined in terms of the cylindrical harmonic waves as the bases and hence \( R_{\nu,v-1} \neq R_{\nu-1,v} \) and \( F_{\nu,v-1} \neq F_{\nu-1,v} \). This situation is different from the reflection and transmission matrices obtained for a planar array [11] of circular rods.

IV. VECTOR-WAVE FORMULATION

The vector-wave problem is formulated by employing \( E_z \) and \( H_z = \sqrt{\mu_0/\varepsilon_0} E_z \) as the leading fields. Other field components can be calculated from \( E_z \) and \( H_z \). Instead of scalar-wave field (1), we define a vector-wave field in the following form:

\[ \bar{\psi}^{(v)} = \begin{bmatrix} E_z^{(v)} \\ H_z^{(v)} \end{bmatrix}. \] (36)

Then the vector-wave field in the \( (v) \)-th region may be expressed as follows:

\[ \bar{\psi}^{(v)} = e^{j\varphi} [\bar{\Phi} \cdot \bar{F}^{(v)} + \bar{\Psi} \cdot \bar{c}^{(v)}] \] (37)

with

\[ \bar{\Phi} = \begin{bmatrix} \Phi^T & 0 \\ 0 & \Psi^T \end{bmatrix}, \quad \bar{\Phi} = \begin{bmatrix} \Phi^T & 0 \\ 0 & \Phi^T \end{bmatrix} \] (38)

\[ \bar{F}^{(v)} = \begin{bmatrix} b_e^{(v)} \\ b_h^{(v)} \end{bmatrix}, \quad \bar{c}^{(v)} = [b_m^{(v)}], \quad b_m^{(v)} = [b_m^{(v)}] \] (39)

\[ \bar{F}^{(v)} = \begin{bmatrix} b_e^{(v)} \\ b_h^{(v)} \end{bmatrix}, \quad \bar{c}^{(v)} = [b_m^{(v)}], \quad b_m^{(v)} = [b_m^{(v)}] \] (39)
\[ \tilde{\mathbf{c}}^{(v)} = \begin{bmatrix} \mathbf{c}^{(v)}_1 \\ \mathbf{c}^{(v)}_2 \end{bmatrix} \quad \mathbf{c}^{(v)}_1 = [e^{(v)}_m], \quad \mathbf{c}^{(v)}_2 = [k^{(v)}_m] \]  
(40)

where \( \{b^{(v)}_m\} \) and \( \{c^{(v)}_m\} \) represent the amplitudes of the incoming standing cylindrical harmonics for \( E_c \) and \( H_c \) fields, respectively, and \( \{e^{(v)}_m\} \) and \( \{k^{(v)}_m\} \) are those of outgoing cylindrical harmonics.

Instead of (7), the vector-wave field scattered by the circular rods on the \( \nu \)-th array is expressed in terms of the local coordinate systems as follows:

\[ \Psi^{sc(v)} = \begin{bmatrix} E_z^{(v)} \\ \hat{H}_z^{(v)} \end{bmatrix} = e^{(v)} \rho \sum_{j=1}^{M} \Psi^{(v)}_j \cdot \mathbf{a}^{(v)} \]  
(41)

with

\[ \Psi^{(v)}_j = \begin{bmatrix} \Psi^{(v)}_j & 0 \\ 0 & \Psi^{(v)}_j \end{bmatrix} \]  
(42)

\[ \mathbf{a}^{(v)} = \begin{bmatrix} a^{(v)}_m \\ a^{(v)}_h \end{bmatrix} \quad a^{(v)}_m = [a^{(v)}_m], \quad a^{(v)}_h = [a^{(v)}_h] \]  
(43)

where \( \{a^{(v)}_m\} \) and \( \{a^{(v)}_h\} \) denote the unknown scattering amplitudes for \( E_c \) and \( H_c \) fields, respectively, and \( \Psi^{(v)}_j \) is defined by (8). The scattered field (41) must satisfy the boundary conditions on the surfaces \( \rho_j = r_j \) \((j=1,2,\ldots,M)\) of each circular rod under the presence of the incident field \( \hat{H}_z^{(v)} \) or \( \Psi^{(v)}_j \cdot \mathbf{a}^{(v)} \). This boundary-value problem is accomplished using the three-dimensional T-matrix of a circular rod which relates the incident \( (E_z, \hat{H}_z) \) field to the scattered one. Following the same analytical procedure as described in Sec. III, a set of linear equations to determine \( \mathbf{a}^{(v)} \) for the incidence of \( \Phi^{T} \cdot \mathbf{a}^{(v)} \) is obtained as follows:

\[ \mathbf{a}^{(v)} = \mathbf{T}^{(v)} \cdot \mathbf{a}^{(v)} \]  
(44)

with

\[ \mathbf{T}^{(v)} = \begin{bmatrix} \mathbf{K}^{(v)} & 0 \\ 0 & \mathbf{K}^{(v)} \end{bmatrix} \quad \mathbf{a}^{(v)} = \begin{bmatrix} a^{(v)}_m \\ 0 \end{bmatrix} \]  
(45)

\[ \mathbf{K}^{(v)} = \begin{bmatrix} k^{(v)} & 0 \\ 0 & k^{(v)} \end{bmatrix} \]  
(46)

where \( \mathbf{K}^{(v)} \) and \( \mathbf{a}^{(v)} \) are defined by (12) and (17), and \( \mathbf{T}^{(v)} \) represents the three-dimensional T-matrix of a circular rod on the \( \nu \)-th layer in isolation which can be obtained in closed form [11]. In the same way, for the incidence of \( \Phi^{T} \cdot \mathbf{a}^{(v)} \) we have the following relation:

\[ \mathbf{a}^{(v)} = \mathbf{T}^{(v)} \cdot \mathbf{a}^{(v)} \]  
(47)

with

\[ \mathbf{T}^{(v)} = \begin{bmatrix} \mathbf{T}^{(v)} \mathbf{K}^{(v)} & 0 \\ 0 & \mathbf{T}^{(v)} \mathbf{K}^{(v)} \end{bmatrix} \]  
(48)

\[ \mathbf{T}^{(v)} = \begin{bmatrix} \gamma^{(v)} & 0 \\ 0 & \gamma^{(v)} \end{bmatrix} \]  
(49)

where \( \gamma^{(v)} \) is defined by (35).

In order to derive the reflection and transmission matrices for the vector-wave field, (41) is transformed to the expression based on the global coordinate \((\rho, \phi)\) as follows:

\[ \Psi^{(v)}_\rho = e^{(v)} \rho \sum_{j=1}^{M} \Psi^{(v)}_j \cdot \mathbf{a}^{(v)} \]  
for \( \rho > R_v \)  
(50)

\[ \Psi^{(v-1)}_\rho = e^{(v)} \rho \sum_{j=1}^{M} \Psi^{(v-1)}_j \cdot \mathbf{a}^{(v-1)} \]  
for \( \rho < R_v \)  
(51)

with

\[ \Psi^{(v)}_\rho = \begin{bmatrix} \chi^{(v)} & 0 \\ 0 & \chi^{(v)} \end{bmatrix} \quad \mathbf{a}^{(v)} = \begin{bmatrix} \eta^{(v)} \\ 0 \end{bmatrix} \]  
(52)

where \( \chi^{(v)} \) and \( \eta^{(v)} \) are defined by (23) and (27). Substituting (44) into (50) and (51) for the incidence of \( \Phi^{(v)} \cdot \mathbf{a}^{(v)} \) field, the reflected field \( \Psi^{(v-1)} \) into the outer region \( (v) \) and the transmitted field \( \Psi^{(v)} \) into the inner region \( (v-1) \) are obtained as follows:

\[ \Psi^{(v)}_\rho = \mathbf{T}^{(v)}_\rho \cdot \mathbf{R}_{v-\rho,v} \cdot \mathbf{a}^{(v)} \]  
(53)

\[ \Psi^{(v-1)}_\rho = \mathbf{T}^{(v-1)}_\rho \cdot \mathbf{R}_{v-\rho,v-1} \cdot \mathbf{a}^{(v-1)} \]  
(54)

\[ \mathbf{R}_{v-\rho,v} = \begin{bmatrix} \chi^{(v)} \mathbf{T}^{(v)}_\rho \\ 0 \end{bmatrix} \]  
(55)

\[ \mathbf{F}_{v-\rho,v-1} = \mathbf{I} + \mathbf{R}_{v-\rho,v} \mathbf{T}^{(v)}_\rho \]  
(56)

where \( \mathbf{R}_{v-\rho,v-1} \) defines the reflection matrix of the \( \nu \)-th cylindrical array which characterizes the reflection from the inner region \( (v-1) \) to the outer region \( (v) \), whereas \( \mathbf{F}_{v-\rho,v} \) defines the transmission matrix of the same array which characterizes the transmission from the outer region \( (v) \) to the inner region \( (v-1) \).

For the case of incidence of \( \Psi^{(v)}_\rho \cdot \mathbf{a}^{(v)} \) field from the inner region \( (v-1) \), on the other hand, the substitution of (47) into (50) and (51) yields the transmitted field \( \Psi^{(v)}_\rho \) into region \( (v) \) and the reflected field \( \Psi^{(v-1)}_\rho \) to region \( (v-1) \) as follows:

\[ \Psi^{(v)}_\rho = \mathbf{T}^{(v)}_\rho \cdot \mathbf{F}_{v-\rho,v-1} \cdot \mathbf{a}^{(v-1)} \]  
(57)
with
\[ \mathbf{F}_{v,v-1} = \mathbf{I} + \mathbf{T}^{(v)} \]

(59)

\[ \mathbf{R}_{v,v-1} = \mathbf{T}^{(v)} \]

(60)

where \( \mathbf{F}_{v,v-1} \) defines the transmission matrix from region \((v-1)\) to region \((v)\) through the \(v\)-th cylindrical array and \( \mathbf{R}_{v,v-1} \) defines the reflection matrix from region \((v)\) to region \((v-1)\). If the cylindrical harmonic expansion in (37) is truncated by the angle of \( \nu \), the positions of the circular rods on the \(v\)-th layer are shifted counterclockwise by an angle \( \theta \) with respect to the global coordinate \(x,O,y\), the reflection and transmission matrices are slightly modified. For the scalar-wave formulation, we have

\[ \mathbf{R}_{x,v-1} = \mathbf{\Omega}_v \mathbf{R}_{x,v-1} \mathbf{\Omega}_v^{-1}, \quad \mathbf{R}_{y,v-1} = \mathbf{\Omega}_v \mathbf{R}_{y,v-1} \mathbf{\Omega}_v^{-1} \]

\[ \mathbf{F}_{x,v-1} = \mathbf{\Omega}_v \mathbf{F}_{x,v-1} \mathbf{\Omega}_v^{-1}, \quad \mathbf{F}_{y,v-1} = \mathbf{\Omega}_v \mathbf{F}_{y,v-1} \mathbf{\Omega}_v^{-1} \]

(61)

(62)

where \( \mathbf{\Omega}_v = [e^{-im\pi\theta}, \delta_{mv}] \) is a diagonal matrix. For the vector-wave formulation, on the other hand, we have

\[ \mathbf{R}_{x,v-1} = \mathbf{\Omega}_v \mathbf{\Omega}_v^{-1}, \quad \mathbf{R}_{y,v-1} = \mathbf{\Omega}_v \mathbf{\Omega}_v^{-1} \]

\[ \mathbf{F}_{x,v-1} = \mathbf{\Omega}_v \mathbf{\Omega}_v^{-1}, \quad \mathbf{F}_{y,v-1} = \mathbf{\Omega}_v \mathbf{\Omega}_v^{-1} \]

(63)

(64)

where

\[ \mathbf{\Omega}_v = \begin{bmatrix} \Omega_v & 0 \\ 0 & \Omega_v \end{bmatrix} \]

(65)

The generalized reflection and transmission matrices of the \(N\)-layered system as shown in Fig. 1 can be obtained by concatenating the reflection and transmission matrices for each layer of the cylindrical arrays. This calculation is performed using \( \mathbf{R}_{v,v-1}, \mathbf{R}_{v,v-1} \), \( \mathbf{F}_{v,v-1}, \mathbf{F}_{v,v-1} \), and \( \mathbf{F}_{v,v-1} \), given by (61)-(64) for the scalar-wave formulation and the vector-wave formulation. Taking into account the scattering process in the layered system, the total field in region \((v)\) is expressed as follows:

\[ \mathbf{\psi}^{(v)} = \mathbf{\psi}^{T} \mathbf{c}^{(v)} + \mathbf{\Phi}^{(v)} \cdot \mathbf{b}^{(v)} \]

(66)

\[ \mathbf{\psi}^{(v)} = (\mathbf{\psi}^{T} \mathbf{\Phi}^{T} \mathbf{\mathbf{R}}_{v,v} \mathbf{c}^{(v)}) \]

(67)

for the scalar-wave formulation and the vector-wave formulation, respectively, where \( \mathbf{\mathbf{R}}_{v,v} \) represents the generalized reflection matrix viewed from region \((v)\) to all of outer regions. The recursive relation for the generalized reflection matrix for the \(N\)-layered structure is obtained \([9][10]\) as

\[ \mathbf{\mathbf{R}}_{v,v} = \mathbf{\mathbf{R}}_{v,v+1} + \mathbf{\mathbf{F}}_{v,v+1} \mathbf{\mathbf{\Lambda}}_{v+1} \mathbf{\mathbf{F}}_{v,v+1}^{-1} \]

(68)

with

\[ \mathbf{\Lambda}_{v+1} = (\mathbf{I} - \mathbf{\mathbf{R}}_{v,v+1}^{-1})^{-1} \]

(69)

Equation (68) is recursively used to calculate the generalized reflection matrices \( \mathbf{\mathbf{R}}_{v,v} \) for the scalar-wave and vector-wave formulations, respectively:

\[ \mathbf{b}^{(1)} = \mathbf{\mathbf{R}}_{1,2} \cdot \mathbf{c}^{(1)}, \quad \mathbf{c}^{(1)} = \mathbf{\mathbf{R}}_{1,0} \cdot \mathbf{b}^{(1)} \]

(70)

\[ \mathbf{\mathbf{b}}^{(1)} = \mathbf{\mathbf{R}}_{1,2} \cdot \mathbf{\mathbf{c}}^{(1)}, \quad \mathbf{\mathbf{c}}^{(1)} = \mathbf{\mathbf{R}}_{1,0} \cdot \mathbf{\mathbf{b}}^{(1)} \]

(71)

Equations (70) and (71) lead to

\[ (\mathbf{I} - \mathbf{\mathbf{R}}_{1,2} \mathbf{\mathbf{R}}_{1,0}) \cdot \mathbf{b}^{(1)} = 0 \]

(72)

\[ (\mathbf{I} - \mathbf{\mathbf{R}}_{1,2} \mathbf{\mathbf{R}}_{1,0}) \cdot \mathbf{\mathbf{c}}^{(1)} = 0 \]

(73)

where \( \mathbf{\mathbf{R}}_{1,2} \) is calculated using the recursive relation (68). Since \( \mathbf{\mathbf{R}}_{1,2} \) and \( \mathbf{\mathbf{R}}_{1,0} \) are the functions of \( \beta \), the mode propagation constant is obtained as the solutions to the dispersion equation

\[ \det[\mathbf{I} - \mathbf{\mathbf{R}}_{1,2}(\beta) \mathbf{\mathbf{R}}_{1,0}(\beta)] = 0 \]

(74)
The solutions to (74) are used in (70) or (71) to determine the amplitude vector $\bar{b}^{(1)}$ and $c^{(1)}(\vec{c}^{(1)})$. Other amplitudes vectors $\bar{b}^{(v)}(\vec{b}^{(v)})$ and $c^{(v)}(\vec{c}^{(v)})$ can be recursively calculated from $\bar{b}^{(1)}(\vec{b}^{(1)})$ and $c^{(1)}(\vec{c}^{(1)})$. For instance, we have the following relations for the scalar-wave case:

$$c^{(0)} = 0, \quad b^{(0)} = \hat{F}_{0,1} \cdot \bar{b}^{(1)}, \quad c^{(1)} = \hat{R}_{1,0} \cdot \bar{b}^{(1)} \quad (75)$$

$$c^{(v)} = \Lambda_v \hat{F}_{v, v+1} \cdot \bar{c}^{(v+1)} \quad (v \geq 2) \quad (76)$$

$$b^{(v)} = \hat{R}_{v, v+1} \cdot \bar{c}^{(v)} \quad (v \geq 2). \quad (77)$$

The same relations as above apply to the vector-wave amplitudes $\bar{b}^{(v)}$ and $\vec{c}^{(v)}$ in the vector-wave formulation.

VII. CONCLUSIONS

We have presented a rigorous semi-analytical approach for a specific microstructured optical fiber, which is formed by layered cylindrical arrays of $M$ circular rods distributed symmetrically on each of $N$-layered concentric circular rings. The method uses the T-matrix of a circular rod in isolation, the reflection and transmission matrices of a cylindrical array, and the generalized reflection and transmission matrices of a cylindrically layered structure. The mode analysis of index-guiding microstructured fibers using the proposed method is under investigation.

REFERENCES


