Cutoff frequencies of electromagnetic waves Propagating in a hexagonal waveguide

Arti Vaish* and Harish Parthasarathy
Division of Electronics and Communication Engineering
Manav Rachna International University,
Faridabad, INDIA. E-mail: vaisharti@gmail.com.

Abstract-In this work, cut-off frequencies of propagation of electromagnetic waves in a hexagonal waveguide are calculated using two-dimensional (2-D) finite element method. The numerical approach is a standard one and involves six finite elements. A new type of hexagonal waveguide structure for the simple homogeneous dielectric case has been considered. The starting point is Maxwell’s equations in conjunction to the exponential dependence of the fields on the Z-coordinates. For the homogeneous case, it results in the Helmholtz equations. Finally, finite element method has been used to derive approximate values of the possible propagation constant for each frequency.

Index Terms- Finite-element-method, Variational principle, Eigenvector, Matrix Equation, frequencies of propagation, hexagonal waveguide.

I. INTRODUCTION

The finite element method (FEM) has been widely used over the decades in the analysis of waveguide components. It is because the propagation characteristics of arbitrarily shaped waveguides of is based on a spatial discretization of cross-section [1]-[5]. This approximation allows handling of waveguide cross section geometries which are very similar to the real structures employed in practice. As a consequence, FEM constitutes a promising tool to characterize such problems [6] - [7]. Modern phased array radars imply the requirements for polarization agility of wideband array elements. Surface hexagonally poled lithium niobate for two dimensional non-linear interactions in optical waveguide structures has been reported [8], [9]. One possible choice for a radiating element with this property is the hexagonal waveguide. In this paper, a numerically efficient finite-element formulation is proposed to solve waveguides problems. Propagation modes obtained by this formulation may be used to analyze problems involving linear systems of arbitrary complex tensor permittivity and permeability. The solution of these eigenvalue problems results in the approximate fields for all components of different eigenmodes in the waveguide which can further be used to obtain the corresponding eigenvalue [10], [11], [12]. A possible comparison of the proposed methodology with the available theoretical results has also been presented herein this paper to claim the accuracy and reliability of the solution method.

Fig. 1. Hexagonal cross section of the waveguide

II. THE FINITE ELEMENT FORMULATION

Maxwell’s equations in free space, no matter what the boundary is, always leads to the Helmholtz equation for any component of the electric or magnetic field, provided that the
region has uniform permittivity and permeability. So the variable \( V(x, y) \) can be treated as the \( z \) component of the electric field inside the hexagonal waveguide and the obvious boundary condition is that this component should vanish on the boundary if we assume that the bounding walls are parallel to the \( z \) axis. The functions \( \Phi(x, y) \) are the test functions which are used for expanding the potential inside each triangular region. They are nothing but interpolation functions. In case the permittivity is non uniform over the waveguide cross section, it can be treated as being piecewise uniform inside each triangular region. The basic idea of taking hexagonal cross-section and dividing the cross-section in to finite number of elements has been taken from elsewhere [2], [6]. In this paper, the equilateral hexagonal cross section of the waveguide is divided into a number of finite elements. An element is considered to be first-order triangular in shape. A schematics of a triangular finite element in the hexagonal waveguide is shown in figure 1. Consider a triangle having vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\). We draw a vector \( u \) joining \((x_1, y_1), (x_2, y_2)\) and another vector \( v \) joining \((x_1, y_1)\) and \((x_3, y_3)\).

\[
d_1 = |u| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\
d_2 = |v| = \sqrt{(x_3-x_1)^2 + (y_3-y_1)^2}
\]

The unit vector along the two directions \( u \) and \( v \) are

\[
\hat{u} = \frac{u}{|u|} = \frac{(x_2-x_1, y_2-y_1)}{d_1} \\
\hat{v} = \frac{v}{|v|} = \frac{(x_3-x_1, y_3-y_1)}{d_2}
\]

any point \((x, y)\) inside this triangle can be represented as

\[
(x-x_1) = \frac{u(x_2-x_1)}{d_1} + \frac{v(x_3-x_1)}{d_2}
\]

Let

\[
(y-y_1) = \frac{u(y_2-y_1)}{d_1} + \frac{v(y_3-y_1)}{d_2}
\]  

Equations (5) and (6) are two linear equations for the variable \( u \) and \( v \) and solving them gives us \( u \), \( v \) as linear functions of \( x \), \( y \). The area measure is given by

\[
ds(u \cdot v) = |\hat{u} \times \hat{v}| \cdot du \cdot dv
\]

Where

\[
|u \times v| = \sin \alpha
\]

here angle \( \alpha \) between the vectors \( u \) and \( v \) defined as

\[
\cos \alpha = \frac{u \cdot v}{d_1 \cdot d_2} = \frac{(x_2-x_1)(x_3-x_1)-(y_2-y_1)(y_3-y_1)}{d_1 \cdot d_2}
\]

and

The integral of a function can be evaluated as

\[
I(\phi) = \frac{1}{2} \int_0^{d_1} \int_0^{d_2} \phi(x_1 + \frac{u(x_2-x_1)}{d_1} + \frac{v(x_3-x_1)}{d_2} + y_1 + \frac{u(y_2-y_1)}{d_1} + \frac{v(y_3-y_1)}{d_2}) \sin(\alpha) du \cdot dv
\]

if \( \Phi = 1 \) then we get

\[
I(\phi) = \frac{d_1 d_2 \sin \alpha}{2}
\]
this is the correct formula for the area of the triangle. Suppose we write

\[ V(x, y) = ax + by + c \]

For \( x, y \in \Delta \)

with \( \Delta \) as the area bounded by the triangle. \( a, b, c \) are chosen so that \( V \) at the vertices are given, i.e.,

\[ V(x_1, y_1) = V_1 \]
\[ V(x_2, y_2) = V_2 \]
\[ V(x_3, y_3) = V_3 \]

Thus,

\[
\begin{pmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  a \\
  b \\
  c \\
\end{pmatrix} =
\begin{pmatrix}
  V_1 \\
  V_2 \\
  V_3 \\
\end{pmatrix}
\]

Thus we find that

\[
a = \frac{V_1(y_2 - y_3) + V_2(y_3 - y_1) + V_3(y_1 - y_2)}{\Delta} \tag{10}
\]
\[
b = \frac{V_1(x_2 - x_3) + V_2(x_3 - x_1) + V_3(x_1 - x_2)}{\Delta} \tag{11}
\]
\[
c = \frac{V_1(x_1y_3 - x_3y_1) + V_2(x_3y_1 - x_1y_3) + V_3(x_1y_2 - x_2y_1)}{\Delta} \tag{13}
\]

where

\[
\Delta = x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1
\]

thus for

\[ x, y \in \Delta \]

we have

\[ V(x, y) = ax + by + c \]

\[ V_1 \phi_1(x, y) + V_2 \phi_2(x, y) + V_3 \phi_3(x, y) \]

\[ \phi_1(x, y) = \frac{(y_2 - y_3)x + (x_2 - x_3)y + (x_2y_3 - x_3y_2)}{\Delta} \tag{14} \]

\[ \phi_2(x, y) = \frac{(y_3 - y_1)x + (x_3 - x_1)y + (x_3y_1 - x_1y_3)}{\Delta} \tag{15} \]

\[ \phi_3(x, y) = \frac{(y_1 - y_2)x + (x_1 - x_2)y + (x_1y_2 - x_2y_1)}{\Delta} \tag{16} \]

The following two integrals occur when one uses the finite element method

First \( I_1 = \int_\Delta V_1^2 \ dx dy \)

Second \( I_2 = \int_\Delta \nabla V_1^2 \ dx dy \)

### TABLE I

<table>
<thead>
<tr>
<th>S.No.</th>
<th>ElementNo.</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Element 1</td>
<td>(0.0, 0.0), (1.0, 1.73), (2.0, 0.0)</td>
</tr>
<tr>
<td>2</td>
<td>Element 2</td>
<td>(0.0, 0.0), (1.0, -1.73), (2.0, 0.0)</td>
</tr>
<tr>
<td>3</td>
<td>Element 3</td>
<td>(0.0, 0.0), (-1.0, -1.73), (1.0, -1.73)</td>
</tr>
<tr>
<td>4</td>
<td>Element 4</td>
<td>(0.0, 0.0), (-2.0, 0.0), (-1.0, -1.73)</td>
</tr>
<tr>
<td>5</td>
<td>Element 5</td>
<td>(0.0, 0.0), (-1.0, 1.73), (-2.0, 0.0)</td>
</tr>
<tr>
<td>6</td>
<td>Element 6</td>
<td>(0.0, 0.0), (-1.0, 1.73), (1.0, 1.73)</td>
</tr>
</tbody>
</table>

Fig. 2. Finite elements mesh (6 elements and 7 nodes). The numbers shown in circles represent the elements.
Now
\[ I_1 = \int_{\Delta} V^2_{(x,y)} \, dx \, dy = \int_{\Delta} (ax + by + c)^2 \, dx \, dy \]
(17)

By substituting the value of \((x, y)\) in terms of \((x_i, y_i)\) in equation (17), we get
\[ I_1 = \int_{\Delta} V^2_{(x,y)} \, dx \, dy \]
\[ = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ a(x_2 - x_1) + \frac{u(x_3 - x_1)}{d_1} \right] \]
\[ + b(y_1 + \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2} + c]^2 \, dudv \]
(18)

The use of method of variable separation for \(u\) and \(v\) results in the following
\[ I_1 = \int_{\Delta} V^2_{(x,y)} \, dx \, dy \]
\[ = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ a(x_2 - x_1) + \frac{u(x_3 - x_1)}{d_1} \right] \]
\[ + v\left(\frac{a(x_3 - x_1) + b(y_3 - y_1)}{d_2} + c'\right)^2 \, dudv \]
(19)

where
\[ c' = ax_1 + by_1 + c \]

Equation (19) can be written as
\[ I_1 = \int_{\Delta} V^2_{(x,y)} \, dx \, dy \]
\[ = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \]
(20)

Here
\[ T_1 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left( a(x_2 - x_1) + \frac{b(y_2 - y_1)}{d_1} \right)^2 \, dudv \]
(21)
\[ T_2 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ a(x_2 - x_1) + \frac{b(y_2 - y_1)}{d_1} \right] \]
\[ \cdot \left[ a(x_3 - x_1) + \frac{b(y_3 - y_1)}{d_2} \right] \, dudv \]
(22)
\[ T_3 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ a(x_2 - x_1) + \frac{b(y_2 - y_1)}{d_1} \right]^2 \, dudv \]
(23)
\[ T_4 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ 2C'(a(x_2 - x_1) + b(y_2 - y_1)) \right] \]
\[ \cdot \left[ a(x_3 - x_1) + \frac{b(y_3 - y_1)}{d_2} \right] \, dudv \]
(24)
\[ T_5 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ 2C'(a(x_2 - x_1) + b(y_2 - y_1)) \right]^2 \, dudv \]
(25)
\[ T_6 = \frac{\sin \alpha}{2} \int_0^{d_1} \int_0^{d_2} \left[ C'_2 \right] \, dudv \]  
(26)

The above integration for first element is given as follows
\[ I_1 = \int_{\Delta} V^2_{(x,y)} \, dx \, dy \]
\[ = .576756v_1^2 + 2.307v_1v_2 + 2.018646v_2^2 \]
\[ + .500077456 \cdot 5v_1(-1.73)v_2 - .500077456 \cdot 5v_1(-1.73)v_3 - 2.018645999 \cdot 8v_2v_3 \]
\[ + .576755999 \cdot 5v_3^2 - 1.7303v_1v_3 \]  
(27)

After calculating the above integrals for each element in the above stated manner, we will find the sum of these integrals over the elements in which we have divided the cross-section. Here we have divided the cross-section into 6 elements (Fig. 2). Summation of these integrals will result in a matrix \(B\) of size \(7 \times 7\).
III. CALCULATION OF INTEGRAL |∇V|²

\[
\int_0^{d_1} \int_0^{d_2} \left| \nabla V \right|^2 \, dudv = \int_0^{d_1} \int_0^{d_2} \left[ \left( \frac{\partial V}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial y} \right)^2 \right] \, dxdy
\]

(28)

Here,
\[ dxdy = Jdudv \]  

(29)

The Jacobian J is given by
\[
J = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix} = \begin{pmatrix}
x_2 - x_1 & x_3 - x_1 \\
d_1 & d_2
\end{pmatrix}
\]

(30)

Now
\[
dxdy = \frac{(x_2 - x_1)(x_3 - x_1)}{d_1d_2} \, dudv
\]

(31)

\[
\int_0^{d_1} \int_0^{d_2} \left| \nabla V \right|^2 \, dudv = \int_0^{d_1} \int_0^{d_2} \left[ \left( \frac{\partial V}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial y} \right)^2 \right] \\
\frac{(x_2 - x_1)(x_3 - x_1)}{d_1d_2} \, dudv
\]

(32)

Here
\[
\left[ \left( \frac{\partial V}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial y} \right)^2 \right] = a^2 + b^2 
\]

(33)

After substituting all the values in above equation and integrate, we get
\[
\int_0^{d_1} \int_0^{d_2} \left| \nabla V \right|^2 \, dudv = 0.577098v_1^2 + 0.5760718862v_1v_2 + 0.577098v_2 - 0.578124(v_1v_3 + v_2v_3 - v_3^2)
\]

(34)

Here \( v_1, v_2, v_3 \cdots v_n \) are the nodal potential. Solution of integration of \( \left| \nabla V \right|^2 \, dudv \) for all the 6 element, computed in same manner will result in a matrix A of size 7 × 7.

IV. FINDING EIGEN VALUES OF THE MATRIX

Now the process of finding eigen values of the matrix is as follows.
\[
(V^T AV - k^2 V^T BV)
\]

when minimized over V gives the quadratic form defined by
\[
\int_0^{\Delta} \left| \nabla V \right|^2 \, dxdy - k^2 \int_0^{\Delta} V^2 \, dxdy
\]

(35)

Here
\[
\delta \int (\nabla \hat{V}, \nabla \hat{V}) \, dxdy = -2 \int_0^{\Delta} \delta V \nabla^2 V \, dxdy
\]

(36)

And
\[
\delta \int V^2 \, dxdy = \int_0^{\Delta} 2V \delta V \, dxdy
\]

(37)

Now, from equation (33)
\[
-2 \int \delta V \nabla^2 V \, dxdy - 2k^2 \int \delta V V \, dxdy = 0
\]

(38)

Or
\[
\int \delta V (\nabla^2 + k^2) V \, dxdy = 0
\]

(39)

\((\nabla^2 + k^2) V \, dxdy = 0\)  

(40)

equation (38) when evaluated approximately using the finite element method gives
\[
AV - k^2 BV = 0
\]

(41)

\((A - k^2 B)V = 0\)  

(42)

\[ \det(A - k^2 B) = 0 \]  

(43)

Here V is the vector of vertex nodal field values. Solution of this matrix will give the eigen values. These eigen values are the propagation frequencies of the waveguide. Using above method we can calculate the propagation frequencies of a waveguide of any type of cross-section.

V. SIMULATION RESULTS

Simulations were carried out using MATLAB and Maple software for a hexagonal waveguide [13]. By the eigen values of the matrix A−k^2 B (Equation 43), values of (propagation constant) has been calculated. The approximate theoretical values of the propagation constant are also calculated by averaging the values of the
permittivity throughout the waveguide cross-section. For a good treatment of waveguides with arbitrary cross section, one can refer to the classic book Classical Electrodynamics by J. D. Jackson [14]. In order to validate the procedure, the computed result is compared with those obtained from the theoretical analysis. Table 2 compares the Eigenvalues of the fundamental frequencies of propagation of the hexagonal waveguide with the theoretical and computed.

Mode of Transverse Electromagnetic wave in a hexagonal waveguide can not be determined in closed form. They can be determining only through the numerical techniques and we have adopted one such technique. To get the yardstick of the comparison we can approximate the mode of hexagonal waveguide by those of rectangular waveguide/ square waveguide having cross-sectional area equal to that of the hexagon that would guarantee same power flow in the hexagonal and square waveguide if the orders of the fields are the same. For a square waveguide the first few modes can be calculated using the standard formula and

$$ \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2} - k^2 $$

is the value of Propagation constant ($\gamma$). Here $k = \omega \sqrt{\mu \varepsilon} $; In general $\gamma = \alpha + j\beta$, where $\alpha$ is attenuation constant and $\beta$ is the propagation constant. If

$$ k^2 = \omega^2 \mu \varepsilon \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 $$

$\gamma = j\beta, \alpha = 0$

Now the phase constant becomes

$$ \beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2} $$

The cutoff frequency is the operating frequency below which attenuation occurs and above which propagation takes place.

The table-2 compares the first few values of $\gamma$ (Propagation constant) of square waveguide with the hexagonal waveguide of same cross section. The theoretical values are obtained by assuming the square waveguide of same area corresponding to the hexagonal waveguide. Ideally to determine the theoretical value we must solve the Helmholtz equation with Dirichlet boundary condition on the hexagonal boundary but this can not be done in closed form so we had utmost make a comparison of square waveguide by using Numerical Technique.

### Table II

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Eigenvalue (This work)</th>
<th>Eigenvalue (Theoretical)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>.451</td>
<td>.460</td>
</tr>
<tr>
<td>2.</td>
<td>.657</td>
<td>.665</td>
</tr>
<tr>
<td>3.</td>
<td>.7079</td>
<td>.765</td>
</tr>
<tr>
<td>4.</td>
<td>1.22666 \times 10^{-8} + 1.439i</td>
<td>1.220</td>
</tr>
<tr>
<td>5.</td>
<td>5 - .1489 \times 10^{-6} + .0006328i</td>
<td>-.149</td>
</tr>
<tr>
<td>6.</td>
<td>-.4513</td>
<td>-.452</td>
</tr>
<tr>
<td>7.</td>
<td>-.657</td>
<td>-.658</td>
</tr>
</tbody>
</table>

### VI. Conclusion

Maxwell’s equations in free space, no matter what the boundary is, always leads to the Helmholtz equation for any component of the electric or magnetic field, provided that the region has uniform permittivity and permeability. In this paper an advantageous finite-element-method for the hexagonal cross-sectional waveguide problem has been developed by which complex propagation characteristics may be obtained for arbitrarily shaped waveguide. Table 2 shows a comparison between the theoretical and practical values. Note that these values are very close to each other. We can see from the table that the results are very close to each other. Attenuation is very low and the propagation is high. We could calculate the modes by using any Numerical technique like method of moments, finite difference method, but the reason for using finite element method here is its simplicity and suitability to the cross-section considered. The extension to higher order elements is straightforward. By suitable modifications of the
method it is possible to treat other types of waveguides as well, e.g. dielectric waveguides with impedance walls and open unbounded dielectric waveguides properly treating the region of infinity.

VII. ACKNOWLEDGEMENT

The authors gratefully acknowledge Prof. Raj Senani, Ar. Rajesh Ayodyawasi and Dr. Ram Prakash Bharti for their constant encouragement and provision of facilities for this research work.

REFERENCES