Scattering of Electromagnetic Plane Waves from Layered Cylindrical Arrays of Circular Rods

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Abstract-A semi-analytical approach for analyzing electromagnetic scattering from layered cylindrical arrays of circular rods is presented. The method uses the T-matrix of a circular rod in isolation, the reflection and transmission matrices for a cylindrical array, and the generalized reflection and transmission matrices for a layered cylindrical system. Numerical examples demonstrate that the layered cylindrical arrays are effective for forming a directed beam in the scattered fields.

Index Terms—scattering, layered cylindrical arrays, circular rods, T-matrix.

I. INTRODUCTION

Periodic dielectric or metallic structures are a subject of continuing interest because of their wide use for practical devices in microwaves and optical waves. A periodic array of circular rods is typical of a discrete periodic structure. The electromagnetic response is characterized by the scattering properties of the individual rods and the multiple interactions among the rods periodically situated. Various analytical or numerical techniques have been developed over the years to formulate the electromagnetic scattering from the periodic arrays [1]-[5]. However, the previous pertinent efforts have been mostly concerned with the planar arrays. An alternative of the array configuration is a cylindrical array which is formed by circular rods periodically arranged on a circular ring. Recently, the cylindrically periodic systems, being referred to as cylindrical electromagnetic bandgap (EBG) structures, have received a growing attention because of their potential applications to the designs of directive antennas or beam-switching antennas. The beam forming properties and the frequency bands of the directive radiation in a cylindrical EBG structure have been reported using FDTD method [6]-[8].

In this paper, we propose a semi-analytical approach for cylindrically periodic arrays formed by circular rods periodically distributed along concentrically layered circular rings. The rods may be dielectric, perfect conductor, air hole or magnetized ferrite. The formulation is based on the T-matrix of a circular rod in isolation [9], the reflection and transmission matrices [10] of a cylindrical array for cylindrical waves as the basis, and the generalized reflection and transmission matrices for a cylindrically layered structure [11]. The proposed method is used to analyze the scattering of a plane wave from layered cylindrical arrays. Numerical results demonstrate that the layered cylindrical arrays are effective for forming a directed beam in the scattered fields.

II. FIELDS EXPRESSIONS

The geometry of cylindrical arrays of circular rods located in a homogeneous background medium with material constants $\varepsilon_0$ and $\mu_0$ is shown in Fig. 1(a). The $M$ number of circular rods of infinite length are symmetrically distributed on each of $N$ concentric circular cylinders with radii $R_{\nu}$ ($\nu = 1, 2, \cdots, N$). The $M$
circular rods should be identical along one ring but those on different rings need not be necessarily identical. The circular ring with radius \( R_n \) is labeled as the \( n \)-th layer. The radius and material parameters of the circular rods on the \( n \)-th layer are denoted by \( r_n \), \( \varepsilon_n \), and \( \mu_n \), respectively. The concentric region within \( R_n + r_n < \rho < R_{n+1} - r_{n+1} \) is labeled as region \((n)\).

We consider the scattering of an electromagnetic plane wave whose direction of incidence is normal to the cylinder axis and forms an angle \( \phi \) with respect to the positive \( x \) axis. The scattering problem is two-dimensional. Let \( \psi \) denote the \( E_z \) field for TM-wave problem and the \( H_z \) field for TE-wave problem.

**A. Incident Field**
The incident field \( \psi^i \) is given in the global coordinate \((\rho, \phi)\) as follows:

\[
\psi^i = e^{i(k_\rho \rho + k_\phi \phi)} = \Phi^i \cdot b^{(N)}
\]

with

\[
k_\rho = k \cos \phi^i, \quad k_\phi = k \sin \phi^i
\]

\[
\Phi = [J_m(k \rho) e^{i \rho p}] \quad \text{and} \quad b^{(N)} = [(i)^m e^{-i \rho p}] .
\]

where \( k = \omega \sqrt{\mu_0 \varepsilon_0} \), \( J_m(k \rho) \) is the Bessel function of \( m \)-th order, vector quantities are defined as the column vectors, and the superscript \( T \) denote the transpose of the indicated vectors.

**B. Scattered Fields in Local Coordinates**
The field scattered by the rods of the \( v \)-th layer is expressed as follows:

\[
\psi^{(v)} = \sum_{j=1}^{M} \sum_{m=-\infty}^{\infty} a_j^{(v)} H_m^{(1)}(k \rho_{v,j}) e^{i m \theta_{v,j}}
\]

where \((\rho_{v,j}, \theta_{v,j})\) \((j = 1, 2, 3, \cdots, M)\) are the local coordinates whose origins are located at the center of each circular rod as shown in Fig. 1(b), \( H_m^{(1)} \) is the \( m \)-th order Hankel function of the first kind, and \( \{a_j^{(v)}\} \) denote the unknown scattering amplitudes. Equation (5) is rewritten in the matrix form as follows:

\[
\psi^{(v)} = \sum_{j=1}^{M} \mathbf{a}_j^{(v)} \cdot \mathbf{H}^{(1)}_{m}(k \rho_{v,j}) e^{i m \theta_{v,j}}
\]

where

\[
\mathbf{a}_j^{(v)} = [H_m^{(1)}(k \rho_{v,j}) e^{i m \theta_{v,j}}] \quad \text{and} \quad \mathbf{H}^{(1)}_{m}(k \rho_{v,j}) = [a_j^{(v)}] .
\]
C. Total Fields in Global Coordinate

The total field in region (v) is expressed as:

$$\psi^{(v)} = \sum_{m=-\infty}^{\infty} \left[ b_m^{(v)} J_m(k\rho) + c_m^{(v)} H_m^{(1)}(k\rho) e^{imp} \right]$$

where \{b_m^{(v)}\} represent the amplitudes of the incoming standing cylindrical waves and \{c_m^{(v)}\} are those of the outgoing cylindrical waves. Equation (8) is rewritten in the matrix form as follows:

$$\psi^{(v)}(\rho, \phi) = \Phi^T \cdot b^{(v)} + \Psi^T \cdot c^{(v)}$$

where

$$\Phi = [H_m^{(1)}(k\rho) e^{imp}]$$

$$b^{(v)} = [b_m^{(v)}]$$

$$c^{(v)} = [c_m^{(v)}]$$

III. REFLECTION AND TRANSMISSION MATRICES

The scattering process characterizing the reflection and transmission through the v-th layer of the arrays is schematically illustrated in Fig. 1(b). Let us assume first that the field \(\Phi^T \cdot b^{(v)}\) is incident from region (v). The scattered field (6) must satisfy the boundary conditions on the surface \(\rho_{x,i} = r_i\) (j = 1, 2, 3, ..., M) of each circular rod under the presence of the incident field \(\Phi^T \cdot b^{(v)}\). To apply the boundary condition on \#i rod, \(\psi^{(v)}(j \neq i)\) and \(\Phi^T \cdot b^{(v)}\) fields are transformed into those expressed by the local coordinate (\(\rho_{x,i}, \phi_{x,i}\)). This transformation is performed by using Graf’s addition theorem [12]. After straightforward manipulations, a set of linear equations to determine the unknown scattering amplitude vectors \(a^{(v)}\) is obtained as follows:

$$\begin{bmatrix} I & A_1^{(v)} & \cdots & A_{M}^{(v)} \\ A_1^{(v)} & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{M}^{(v)} & \cdots & \cdots & I \end{bmatrix} \begin{bmatrix} a_1^{(v)} \\ a_2^{(v)} \\ \vdots \\ a_{M}^{(v)} \end{bmatrix} = \begin{bmatrix} T^{(v)} a_1^{(v)} - b^{(v)} \\ T^{(v)} a_2^{(v)} - b^{(v)} \\ \vdots \\ T^{(v)} a_{M}^{(v)} - b^{(v)} \end{bmatrix}$$

with

$$A_p^{(v)} = -T^{(v)}K_p^{(v)} \quad (p = 2, 3, \ldots, M)$$

$$K_p^{(v)} = [H_n^{(1)}(kd_{v,p})e^{i(\delta_{vp})}]$$

$$a_q^{(v)} = [(-1)^{q-n} J_n(kR_s)e^{i(\delta_{qv})}]$$

$$b^{(v)} = [b_m^{(v)}]$$

$$\delta_{vp} = \frac{\pi}{2} - (p-1)\frac{\theta_{il}}{2} \quad (p = 2, 3, \ldots, M)$$

$$d_{v,p} = 2R_s |(p-1)\theta_{il}/2|, \theta_{il} = 2\pi/M$$

where \(T^{(v)}\) is the T-matrix of a single circular rod located on the v-th circular ring, which is obtained in closed form [9]. \(a_q^{(v)} (q = 1, 2, \ldots, M)\) is the translation matrix of the singular part of cylindrical waves from the global coordinate (\(\rho, \phi\)) to the \#q local coordinate (\(\rho_{x,q}, \phi_{x,q}\)). \(K_p^{(v)}\) is the translation matrix of the singular part of cylindrical waves from the \#p local coordinate (\(\rho_{x,p}, \phi_{x,p}\)) to the \#1 local coordinate (\(\rho_{x,1}, \phi_{x,1}\)), and \(d_{v,p}\) is the distance between the centers of the \#p and \#1 rods.

Since the linear equation (13) is formed by a block circulant matrix, the solutions to \(a_j^{(v)}\) is obtained in closed form using the eigenvalues and eigenvectors of the \(M \times M\) circulant matrix. From a straightforward manipulation, we have [13]

$$a_j^{(v)} = \overline{T}_j^{(v)} \cdot b^{(v)}$$

with

$$\overline{T}_j^{(v)} = \sum_{q=1}^{M} \sigma_q^{(v)} T^{(v)} a_q^{(v)} \Theta^{-1} \quad (j = 2, 3, \ldots, M)$$

$$\sigma_q^{(v)} = \sum_{l=1}^{M} e^{-i(l-1)\delta_{qv}} (\Lambda_l^{(v)})^{-1}$$

$$\Lambda_l^{(v)} = I + \sum_{p=2}^{M} e^{i(l-1)(p-1)\delta_{vp}} A_p^{(v)}$$

$$\Theta = [e^{i\text{det} \delta_{mu}}]$$

Next, let us consider that the outgoing cylindrical wave \(\psi^{(v)} \cdot c^{(v-1)}\) is incident on the cylindrical array from the inner region (v-1). Following the same analytical procedure as mentioned above, the local scattering amplitude vectors \(a^{(v)}\) are obtained as follows:

$$a_j^{(v)} = \overline{T}_j^{(v)} \cdot c_{j-1}^{(v-1)}$$

with
with

Using Eq.(20) in Eq.(6), the scattered field is expressed in terms of the amplitude vector \( \mathbf{b}^{(v)} \) of the incident field. As the next step to obtain the reflection and transmission matrices, Eq.(6) is transformed into the expression based on the global coordinate \((\rho, \phi)\) by making use of Graf's addition theorem [12]. The expressions are different in region \((v)\) with \(\rho > R_v\) and in region \((v-1)\) with \(\rho < R_v\). When \(\rho > R_v\), we have:

\[
\psi^{(v)} = \Phi^T \cdot \sum_{j=1}^{M} \mathbf{b}_j^{(v)} \cdot \mathbf{a}_j^{(v)}
\]  

where

\[
\mathbf{b}_j^{(v)} = [J_{m-\nu}(kR_v) \cdot e^{-j(m-1)\theta_u}] .
\]  

Substituting Eq.(20) into Eq.(29), the reflected field into region \((v)\) is obtained as follows:

\[
\psi^{(v)} = \Phi^T \cdot \mathbf{R}_{v,v-1} \cdot \mathbf{b}^{(v)}
\]  

with

\[
\mathbf{R}_{v,v-1} = \sum_{j=1}^{M} \mathbf{b}_j^{(v)} \overline{T}_{v,j}^{(v)}
\]  

where \(\mathbf{R}_{v,v-1}\) defines the reflection matrix of the \(v\)-th layer, which characterizes the reflection from region \((v)\) to region \((v-1)\). Note that the unit matrix \(\mathbf{I}\) contained in Eq.(36) indicates the contribution of the incident field \(\Phi^T \cdot \mathbf{b}^{(v)}\) in region \((v-1)\).

In the same way, Eq.(25) is used in Eqs.(29) and (33) for the incidence of \(\Psi^T \cdot \mathbf{e}^{(v-1)}\), and the transmitted field \(\psi^{(v)}\) into region \((v)\) and the reflected field \(\psi^{(v-1)}\) into region \((v-1)\) are obtained as follows:

\[
\psi^{(v)} = \Psi^T \cdot \mathbf{F}_{v,v-1} \cdot \mathbf{e}^{(v-1)}
\]  

\[
\psi^{(v-1)} = \Phi^T \cdot \mathbf{R}_{v,v-1} \cdot \mathbf{e}^{(v-1)}
\]  

with

\[
\mathbf{F}_{v,v-1} = \mathbf{I} + \sum_{j=1}^{M} \eta_j^{(v)} \overline{T}_{v,j}^{(v)}
\]  

\[
\mathbf{R}_{v,v-1} = \sum_{j=1}^{M} \eta_j^{(v)} \overline{T}_{v,j}^{(v)}
\]  

where \(\mathbf{F}_{v,v-1}\) defines the transmission matrix from region \((v-1)\) to region \((v)\) through the \(v\)-th layer of the arrays and \(\mathbf{R}_{v,v-1}\) defines the reflection matrix from region \((v)\) to region \((v-1)\).

### IV. GENERALIZED REFLECTION AND TRANSMISSION MATRICES

When the center of #1 rod is located on the \(x\) axis as shown in Fig. 1(b), the reflection and transmission matrices \(\mathbf{R}_{x,y,1}, \mathbf{F}_{x,y,1}, \mathbf{F}_{y,x,1}\), and \(\mathbf{R}_{x,y,1}\) for the \(v\)-th layer of the arrays are given by (32), (36), (39), and (40), respectively. If the positions of \(M\) rods on the \(v\)-th layer are rotated counterclockwise by an angle of \(\delta_v\) with respect to the global coordinate \(x-O-y\), these reflection and transmission matrices are slightly modified as follows:

\[
\hat{\mathbf{R}}_{x,y,1} = \Omega_y \mathbf{R}_{x,y,1} \Omega_y^{-1}, \quad \hat{\mathbf{F}}_{x,y,1} = \Omega_y \mathbf{F}_{x,y,1} \Omega_y^{-1}
\]  

\[
\hat{\mathbf{F}}_{y,x,1} = \Omega_x \mathbf{F}_{y,x,1} \Omega_x^{-1}, \quad \hat{\mathbf{F}}_{y,x,1} = \Omega_x \mathbf{F}_{y,x,1} \Omega_x^{-1}
\]  

with
\[ \Omega_r = [e^{-im\delta_r}] \]  \hspace{1cm} (43)

where \( \Omega_r \) is a diagonal matrix which corrects the phase of each cylindrical waves due to the offset of the circular rods.

The generalized reflection and transmission matrices of the \( N \)-layered system as shown in Fig. 1(a) can be obtained by concatenating the reflection and transmission matrices \( \bar{R}_{r-1} \), \( \bar{R}_{r-1} \), \( \tilde{F}_{r-1} \), and \( \tilde{F}_{r-1} \) for each layer of the cylindrical arrays. By tracing the scattering processes shown in Fig. 1(b) over the layered cylindrical arrays, a recurrence relation for the generalized reflection matrix \( \bar{R}_{r-1} \) viewed from region \( (r) \) to all of the inner regions is obtained as follows [10],[11]:

\[ \bar{R}_{r-1} = \bar{R}_{r-1} + \tilde{F}_{r-1} \left( I - \bar{R}_{r-1} \right) \tilde{F}_{r-1} \]  \hspace{1cm} (44)

where \( \tilde{F}_{i} = 0 \). Equation (44) can be also applied to the array configuration in which an isolated single circular rod is additionally located at the global origin \( O \) of Fig.1(a). In this case, we may start the recursive relation (44) from \( n = 0 \) using the initial condition

\[ \bar{R}_{0-1} = T^{(0)}, \quad \tilde{F}_{0-1} = 0 \]  \hspace{1cm} (45)

where \( T^{(0)} \) denotes the T-matrix of the circular rod located in the global origin.

In the outermost region \( (N) \), the scattered field is expressed as follows:

\[ \psi^{(N)} = \Psi^{(0)} \tilde{R}_{N,N-1} \cdot b^{(N)} \]  \hspace{1cm} (46)

where \( b^{(N)} \) denotes the amplitude vector defined by Eq.(4) for the incident plane wave. \( \bar{R}_{N,N-1} \) is calculated recursively by using Eq.(44). The scattering cross section and scattering pattern can be easily calculated from Eq.(46) by using the asymptotic behavior of \( H^{(1)}_m(k \rho) \) for \( k \rho \gg 1 \).

V. NUMERICAL RESULTS

We have used the proposed method to analyze the differential scattering cross sections of the layered cylindrical arrays. Although a substantial number of numerical results could be generated, we demonstrate here the results of a three-layered structure for the incidence of the TM wave as shown in Fig. 2. Four dielectric circular rods are periodically distributed along each of three circular rings with offset angles \( \delta_1 = 0^\circ \) and \( \delta_2 = 45^\circ \) respectively.

For numerical calculations, the T-matrix of a circular rod and the translation matrices associated with Graf's addition theorem were truncated up to \( m = n = \pm 24 \). In this case, the size of the reflection and transmission matrices used in the computation becomes 49x49 dimensions. The accuracy of the numerical solutions is validated through the test for reciprocity relation and optical theorem [14].

Normalized scattering patterns of three different array configurations calculated for the incidence of TM wave with (a) \( \phi^i = 0^\circ \) and (b) \( \phi^i = 45^\circ \) are plotted in Fig. 3, where \( R_1 = 1.5 \lambda \), \( R_2 = 1.5 \lambda \), \( R_3 = 2 \lambda \), \( \eta_1 = 0.1 \lambda \), \( \eta_2 = 1.5 \lambda \), \( \eta_3 = 2 \lambda \), and \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 5.0 \varepsilon_0 \). Note that the radius of circular rods increases towards the outer layers. In Fig. 3, (a-1) and (b-1) show the results for 1-layer structure, (a-2) and (b-2) are those of 2-layers structure, and (a-3) and (b-3) are those of
3-layers structure. We can see that the scattering pattern of 1-layer structure is substantially changed by stacking additional cylindrical layers. For 3-layers structure, the scattering in the forward direction is enhanced and a very sharp directed beam is formed in this direction.

Fig. 3. Normalized scattering patterns of three different configurations of cylindrical arrays with \( M = 4 \) for the incidence of TM wave with (a) \( \varphi^i = 0^\circ \) and (b) \( \varphi^i = 45^\circ \), where (a-1) and (b-1) are 1-layer structure with \( R_1 = 1.5 \lambda, \eta = 0.1 \lambda, \varepsilon_1 = 5.0 \varepsilon_0, \) and \( \delta_1 = 0^\circ \). (a-2) and (b-2) are 2-layers structure with \( R_1 = 1.5 \lambda, R_2 = 1.5 R_1, \eta = 0.1 \lambda, \rho_2 = 1.5 \eta, \varepsilon_1 = \varepsilon_2 = 5.0 \varepsilon_0, \delta_1 = 0^\circ, \) and \( \delta_2 = 45^\circ \). and (a-3) and (b-3) are 3-layers structure with \( R_1 = 1.5 \lambda, R_2 = 1.5 R_1, R_3 = 2 R_1, \eta = 0.1 \lambda, \rho_2 = 1.5 \eta, \rho_3 = 2 \eta, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 5.0 \varepsilon_0, \delta_1 = \delta_2 = 0^\circ, \) and \( \delta_3 = 45^\circ \).
VI. CONCLUSION

We have proposed a semi-analytical approach for two-dimensional electromagnetic scattering from layered cylindrical arrays, which are formed by circular rods periodically distributed on each of concentric circular rings. The method uses the T-matrix of a circular rod in isolation, the reflection and transmission matrices of a cylindrical array, and the generalized reflection and transmission matrices for a layered cylindrical structure. The only approximation involved in the method is the truncation of the T-matrix and the translation matrices for cylindrical waves, which is necessary to solve the linear equations for the unknown by matrix inversion. We have used the method to analyse the scattering of a TM plane wave by three-layered cylindrical arrays of dielectric circular rods with tapered radii in the radial direction. From numerical examples, it was shown that if the circular rods are properly arranged, the layered cylindrical arrays are very effective for refining and focusing the scattered field in the forward direction.

Although the present formulation was devoted to the two-dimensional field, the extension to the vector field excited by a plane wave of oblique incidence is straightforward. In this case, we must introduce two scalar fields representing $E_z$ and $H_z$ fields. The T-matrix of a circular rod, the reflection and transmission matrices, and the generalized reflection and transmission matrices are defined in terms of the cylindrical waves of both $E_z$ and $H_z$. Then the size of matrices to be solved becomes four times larger than that of the present two-dimensional case.

REFERENCES